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Research Article

Bounds on the largest eigenvalue and energy of the connectivity matrix

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Abstract

The connectivity matrix of a graph is the matrix whose off-diagonal i-j entry is the maximum number of internally vertex disjoint paths between the vertices i and j with only zeros on its diagonal. In this article, lower and upper estimates for the largest eigenvalue of such matrices are investigated. Bounds in terms of the underlying graph's average degree are provided and it is shown that they can be utilized to refine previous bounds on the energy of connectivity matrices.

Keywords: graph eigenvalues; graph energy; independent paths; vertex cuts; connectivity matrices; path matrices.

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1. Introduction

Throughout this article, graphs are nonempty, finite, undirected, and contain neither loops nor multiple edges. The terms and concepts not introduced here explicitly are taken from the monograph of Diestel [5]. The identity matrix is denoted by I. A vector, of appropriate size, that contains only ones is denoted by $\mathbb{1}$. For a graph G on vertex set $V(G) = \{1, \ldots, n\}$ its *order* and *size* are abbreviated by $n := |V(G)|$ and $m := |E(G)|$, respectively. Denote the *connectivity* of G by κ, its *degrees* by $\Delta = d_1 \geq d_2 \geq \ldots \geq d_n = \delta$, and its *average degree* by $\bar{d} := \frac{1}{n}\sum_{i=1} d_i = \frac{2m}{n}$.

Our main interest here is in *connectivity matrices*, also known as *path matrices*. For a graph G, this is a matrix $P(G) = [p_{ij}]$ of type $n \times n$, whose diagonal entries are zero and whose off-diagonal entry p_{ij} is the maximum number of *independent*, meaning internally vertex disjoint, paths between vertices i and j in G. The number p_{ij} is also called *local connectivity* of i and j and graphs satisfying $p_{ij} = \min\{d_i, d_j\}$ for all vertices i and j are called *maximally local connected*. Similar to a graph's adjacency matrix, the connectivity matrix resembles how different vertices are connected. But the connectivity matrix has very special weights and contains non-zero off-diagonal entries only if the underlying graph is not connected. Questions about how the spectral properties of connectivity matrices are related to the underlying graph's structure are studied first by Patekar and Shikare [14]. Naturally, the largest eigenvalue of $P(G)$, which we address by ρ_1, received attention. It is known that

$$\kappa(n-1) \leq \rho_1 \leq \Delta(n-1).$$

The lower bound is established by Akbari, Ghodrati, Gutman, Hosseinzadeh, and Konstantinova [1], the upper bound is given by Narke and Malavadkar [13]. Both results can be shown by employing Perron-Frobenius arguments, as presented by Horn and Johnson [8, Chapter 8]. Refined upper bounds on ρ_1 are given by Xu and Zhou [18]. Those are in terms of vertex transmissions or the trace of $P(G)^2$, but so far we lack simple bounds in terms of the average degree. In comparison, the classical relations by Collatz and Sinogowitz [3] for the largest adjacency eigenvalue λ_1 are

$$\bar{d} \leq \lambda_1 \leq \Delta.$$

Encouraged by this blueprint, we ask for bounds on ρ_1 that depend on $\bar{d}$, instead of κ or Δ, in Section 2. As $\kappa \leq \delta \leq \bar{d} \leq \Delta$, such relations are a good complement to previous results. In Section 3, we demonstrate the potential of such eigenvalue bounds in showing how to obtain upper bounds on the energy of connectivity matrices that improve existing ones.

2. Bounds on the largest eigenvalue

First note that the Rayleigh principle implies the simple relation

$$\rho_1 \geq \frac{1}{n}\mathbb{1}^\top P(G)\mathbb{1} \geq \frac{2m}{n} = \bar{d}.$$

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To improve on that, let us recall key facts about local connectivity, presented by Volkmann [17] and Mader [12].

Lemma 2.1 (see [17])**.** *For a graph G of order n and two vertices $i, j \in V(G)$, the local connectivity p_{ij} satisfies*

$$p_{ij} \geq d_i + d_j - n + [ij \in E(G)] + 2[ij \notin E(G)].$$

Herein, we use the Iverson bracket, which evaluates to $[B] := 1$ if B is true and $[B] := 0$ if B is false.

Theorem 2.1 (see [12])**.** *Any graph G that has at least one edge contains two adjacent vertices $i, j \in V(G)$ that are connected by $\min\{d_i, d_j\}$ independent paths.*

Theorem 2.2. *Consider a graph G on n vertices whose connectivity matrix has the largest eigenvalue ρ_1. Then*

$$(2n-3)\bar{d} + (n-1)(2-n) \leq \rho_1,$$

with equality if and only if G is complete, or consists of two isolated vertices, or is a path on three vertices.

Proof. By the Rayleigh principle and Lemma 2.1, we obtain

$$\begin{aligned}
\rho_1 &\geq \frac{1}{n}\mathbb{1}^\top P(G)\mathbb{1} = \frac{2}{n}\sum_{1\leq i<j\leq n} p_{ij} \\
&\geq \frac{2}{n}\sum_{1\leq i<j\leq n}\Big(d_i + d_j - n + [ij \in E(G)] + 2[ij \notin E(G)]\Big) \\
&= \frac{2}{n}\Big((n-1)\sum_{i=1}^{n} d_i - \frac{n^2(n-1)}{2} + m + 2\Big(\frac{n(n-1)}{2} - m\Big)\Big) \\
&= (2n-2)\bar{d} - n(n-1) + 2(n-1) - \frac{2m}{n} \\
&= (2n-3)\bar{d} + (n-1)(2-n),
\end{aligned}$$

which verifies the claimed bound. The relation is tight if and only if $p_{ij} = d_i + d_j - n + [ij \in E(G)] + 2[ij \notin E(G)]$ for all vertices $i \neq j$ in $V(G)$ and if $\mathbb{1}$ is an eigenvector to ρ_1. Then for any $s \in V(G)$ row s of $P\mathbb{1}$ reads

$$\begin{aligned}
\sum_{j=1}^{n} p_{sj} &= \sum_{j=1,\, j\neq s}^{n}\Big(d_s + d_j - n + [sj \in E(G)] + 2[sj \notin E(G)]\Big) \\
&= (n-1)d_s + \sum_{j=1,\, j\neq s}^{n} d_j - n(n-1) + d_s + 2(n-1-d_s) \\
&= (n-3)d_s + 2m + (2-n)(n-1).
\end{aligned}$$

The same argument can be made for any other vertex $t \neq s$ in $V(G)$. From $P\mathbb{1} = \rho_1\mathbb{1}$, we obtain

$$(n-3)d_s + 2m + (2-n)(n-1) = \rho_1 = (n-3)d_t + 2m + (2-n)(n-1).$$

For $n \neq 3$, this implies $d_s = d_t$. In other words, only graphs on three vertices or regular graphs can possibly attain the given bound. It is easy to see that for $n = 3$, only the path on three vertices and the triangle attain our bound. Finally, let G be a k-regular graph attaining our bound. By Theorem 2.1, there are two adjacent vertices $v, w \in V(G)$ that are connected by $\min\{d_v, d_w\}$ independent paths in G. This implies

$$k = \min\{d_v, d_w\} = p_{vw} = d_v + d_w - n + [vw \in E(G)] + 2[vw \notin E(G)] = 2k - n + 1.$$

Consequently, $k = n-1$. So G is complete, as claimed. Note that Mader's result [12] is not applicable if G is edgeless. The only such graph attaining our bound is the graph consisting of two isolated vertices. □

Another bound on ρ_1 can be deduced directly from a result of Dankelmann and Oellermann [4]. Whereas the authors provide graphs that attain the given bound, we contribute an argument that the given candidates are the only attaining graphs. Before we reinvestigate their proof, let us recall Dirac's characterization [6] of *chordal* graphs, meaning graphs that contain no cycle on four vertices as an induced subgraph.

Theorem 2.3 (see [6])**.** *A graph G is chordal if and only if every induced subgraph of G contains a* simplicial *vertex, that is a vertex whose neighborhood is complete.*

Corollary 2.1. *Any $(k-1)$-regular chordal graph is of the form cK_k, for some $c \in \mathbb{N}$, meaning a disjoint union of complete graphs on k vertices.*

Theorem 2.4 (see [4]). *For a graph G of order n, there holds*

$$\bar{\kappa}(G) \geq \frac{\bar{d}^2}{n-1},$$

where $\bar{\kappa}(G) := \binom{n}{2}^{-1} \sum_{1 \leq i < j \leq n} p_{ij}$ denotes the average connectivity *of G. The bound is attained if and only if G is of the form $\frac{n}{k}K_k$, where n has to be a multiple of k.*

Proof. Let us denote the number of i-j paths of length $\ell \in \{1,2\}$ by $p_{\ell\, ij}$. Note that paths counted by these coefficients are always independent. Furthermore,

$$\sum_{1 \leq i < j \leq n} p_{1\,ij} = E(G) = \frac{n\bar{d}}{2}.$$

Because every vertex i of G is the inner vertex of $\binom{d_i}{2}$ paths of length two, there holds

$$\sum_{1 \leq i < j \leq n} p_{2\,ij} = \sum_{1 \leq i < j \leq n} \binom{d_i}{2} = \sum_{1 \leq i < j \leq n} \frac{d_i(d_i-1)}{2} \geq \frac{n\bar{d}(\bar{d}-1)}{2},$$

by the convexity of the function $f(x) = x(x-1)/2$. Herein, equality holds if and only if G is regular. Consequently,

$$\bar{\kappa}(G) = \binom{n}{2}^{-1} \sum_{1 \leq i < j \leq n} p_{ij} \geq \frac{2}{n(n-1)} \sum_{1 \leq i < j \leq n} (p_{1\,ij} + p_{2\,ij}) \geq \frac{2}{n(n-1)} \left(\frac{n\bar{d}}{2} + \frac{n\bar{d}(\bar{d}-1)}{2} \right) = \frac{\bar{d}^2}{n-1}.$$

This inequality is attained if and only if G is regular and if $p_{ij} = p_{1\,ij} + p_{2\,ij}$ for all i and j. The latter condition can only be satisfied if G contains no induced cycle on at least four vertices, in other words, if G is chordal. In view of Theorem 2.3 and Corollary 2.1, this concludes our proof. □

Corollary 2.2. *Consider a graph G on n vertices whose connectivity matrix has the largest eigenvalue ρ_1. Then $\bar{d}^2 \leq \rho_1$. The bound is attained if and only if G is of the form $\frac{n}{k}K_k$, where n has to be a multiple of k.*

Proof. By the Rayleigh principle and Theorem 2.1, we obtain

$$\rho_1 \geq \frac{1}{n}\mathbb{1}^\top P(G)\mathbb{1} = \frac{2}{n} \sum_{1 \leq i < j \leq n} p_{ij} = (n-1)\bar{\kappa}(G) \geq \bar{d}^2.$$

The case of equality follows from that of Theorem 2.4 and the fact that $\rho_1 = \frac{1}{n}\mathbb{1}^\top P(G)\mathbb{1}$ if G is of the form $\frac{n}{k}K_k$, where n is a multiple of k. □

We now turn to an upper bound on ρ_1 that depends on the average degree. We rely on the following well-known facts, which can be found in [16, Section 1.3].

Lemma 2.2. *Let $A = [a_{ij}]$ and $B = [b_{ij}]$ be nonnegative irreducible matrices in $\mathbb{R}^{n \times n}$. If $A \leq B$, then $\lambda_1(A) \leq \lambda_1(B)$. If $A \leq B$ and if $a_{ij} < b_{ij}$ for some $i, j \in \{1, \dots, n\}$, then $\lambda_1(A) < \lambda_1(B)$.*

Lemma 2.3. *The largest eigenvalue λ_1 of any nonnegative symmetric matrix $A = [a_{ij}]$ in $\mathbb{R}^{n \times n}$ satisfies*

$$\lambda_1 \leq \max_{1 \leq i \leq n} \sum_{j=1}^{n} a_{ij}$$

Theorem 2.5. *Consider a graph G on n vertices whose connectivity matrix has the largest eigenvalue ρ_1. Then $\rho_1 \leq (n-1)\bar{d}$. Herein, equality holds if and only if G is regular and maximally local connected.*

Proof. Recall that we denote the degrees of G such that $d_1 \geq \ldots \geq d_n$. By the fact that $p_{ij} \leq \min\{d_i, d_j\}$, we obtain

$$P(G) \leq \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ \vdots \\ n-1 \\ n \end{array} \begin{array}{c} \begin{array}{cccccc} 1 & 2 & 3 & \cdots\cdots & n-1 & n \end{array} \\ \left[\begin{array}{cccccc} 0 & d_2 & d_3 & \cdots\cdots & d_{n-1} & d_n \\ d_2 & 0 & d_3 & \ddots & \vdots & d_n \\ d_3 & d_3 & 0 & \ddots & \vdots & d_n \\ \vdots & & \ddots & 0 & d_{n-1} & d_n \\ d_{n-1} & \cdots & \cdots & d_{n-1} & 0 & d_n \\ d_n & d_n & d_n & d_n & d_n & 0 \end{array}\right] \end{array} =: D(G).$$

The first row of $D(G)$ is that with the largest row sum. By Lemmas 2.2 and 2.3, we conclude that

$$\rho_1 \le \sum_{i=2}^{n} d_i = \frac{\sum_{i=2}^{n} d_i + (n-1)\sum_{i=2}^{n} d_i}{n} \le \frac{(n-1)d_1 + (n-1)\sum_{i=2}^{n} d_i}{n} = (n-1)\frac{\sum_{i=1}^{n} d_i}{n} = (n-1)\bar{d}. \quad (1)$$

In view of Lemma 2.2, our bound can only be attained if $P(G) = D(G)$, that is, if G is maximally local connected. The second inequality in (1) is attained if and only if G is regular. But then $P(G) = \bar{d}(\mathbb{1}\mathbb{1}^\top - I)$ and thus $P(G)\mathbb{1} = (n-1)\bar{d}\mathbb{1}$ and our bound is attained. □

3. Upper bounds on the energy

Let us recall that the *energy* $\mathcal{E}(A)$ *of a matrix* A is defined as the sum of the absolute values of the eigenvalues of A. By the *energy of a graph*, one typically refers to the energy of its associated adjacency matrix. Other graph matrices give rise to further variants of that concept. The *connectivity* or *path energy* is $\mathcal{E}_P(G) := \mathcal{E}(P(G))$. This particular invariant is introduced by Shikare, Malavadkar, Patekar, and Gutman [15], who also asked whether the following relation holds.

Conjecture 3.1 (see [15]). *For a graph G of order n, its connectivity energy satisfies*

$$\mathcal{E}_P(G) \le 2(n-1)^2,$$

with equality if and only if G is complete.

If true, this relation shows an interesting difference between the energy of connectivity and adjacency matrices. The latter is not maximal for the complete graph, which led to extensive research on those graphs whose energy exceeds the energy of complete graphs, so-called *hyperenergetic* graphs. But in our case, it is not known whether there exist such candidates at all. See also Gutman [11] for an overview about that topic. Attempts to resolve Conjecture 3.1 are made by Ilić and Bašić [9] and Hofmann [7], who verify that

$$\mathcal{E}_P(G) \le \Big(2n\sum_{i=1}^{n}(i-1)d_i^2\Big)^{1/2} \le n(n-1)^{3/2},$$

We conclude this article by illustrating how to improve this bound. Our general strategy follows the approach by Koolen and Moulten [10]. Furthermore, we build on Theorem 2.2 and make use of the following fact about local connectivities, presented by Casablanca, Mol, and Oellermann [2] about the *potential* of a sequence of positive integers $a_1 \ldots, a_n$, that is

$$P(a_1, a_2, \ldots, a_n) := \sum_{1\le i<j\le n} \min\{a_i, a_j\}.$$

Lemma 3.1 (see [2]). *Let $a_1, a_2, \ldots, a_n$ be a sequence of positive integers and set $s := \sum_{i=1}^{n} a_i$. Denoting the unique integers a and r that satisfy $0 \le r < n$ and $s = an + r$, there holds*

$$P(a_1, a_2, \ldots, a_n) \le P(\underbrace{a, \ldots, a}_{n-r \text{ terms}}, \underbrace{a+1, \ldots, a+1}_{r \text{ terms}}).$$

Theorem 3.1. *For a graph G on n vertices, the corresponding connectivity energy satisfies*

$$\mathcal{E}_P(G) \le \rho_1 + \Big((n-1)\big(n(n-1)(\rho_1 + \mathrm{var}(G)) - (n-r)r - \rho_1^2\big)\Big)^{1/2}$$

where $\mathrm{var}(G) := \frac{1}{n}\sum_{i=1}^{n}(d_i - \bar{d})^2$ *is the* degree variance *of G and r is the unique integer such that $0 \le r < n$, $a \in \mathbb{Z}$, and $\sum_{i=1}^{n} d_i^2 = an + r$. Equality holds if and only if $G = K_n$.*

Proof. Let us denote the eigenvalues of $P(G)$ by $\rho_1 \ge \ldots \ge \rho_n$ and recall that ρ_1 is positive. Then, by the Cauchy-Schwarz inequality,

$$\mathcal{E}_P(G) = \rho_1 + \sum_{i=2}^{n}\rho_i \le \rho_1 + \Big((n-1)\sum_{i=2}^{n}\rho_i^2\Big)^{1/2} = \rho_1 + \Big((n-1)(\mathrm{tr}(P^2) - \rho_1^2)\Big)^{1/2}.$$

It remains to be shown a suitable bound on $\mathrm{tr}(P^2)$. For that purpose, we use Lemma 3.1 for the setting $a_i := d_i^2, i \in \{1, \ldots, n\}$. Denote $s := \sum_{i=1}^{n} a_i = \sum_{i=1}^{n} d_i^2$ and let a and r be the unique integers such that $0 \le r < n$ and $s = an + r$. Then

$$a = \frac{1}{n}(s-r) = \frac{1}{n}\Big(\sum_{i=1}^{n} d_i^2 - r\Big).$$

It follows that

$$\begin{aligned}\mathrm{tr}(P^2) &= 2\sum_{1\le i<j\le n} p_{ij}^2 \le 2\sum_{1\le i<j\le n}\min\{d_i,d_j\}^2 = 2\sum_{1\le i<j\le n}\min\{d_i^2,d_j^2\} = 2P(d_1^2,d_2^2,\dots,d_n^2)\\ &\le 2P(\underbrace{a,\dots,a}_{n-r\text{ terms}},\underbrace{a+1,\dots,a+1}_{r\text{ terms}}) = 2\left(a\binom{n}{2}+\binom{r}{2}\right) = an(n-1)+r(r-1)\\ &= \frac{1}{n}\Big(\sum_{i=1}^n d_i^2 - r\Big)n(n-1)+r(r-1) = n(n-1)\Big(\Big(\frac{1}{n}\sum_{i=1}^n d_i\Big)^2+\frac{1}{n}\sum_{i=1}^n(d_i-\bar d)^2\Big)-(n-r)r\\ &= n(n-1)(\bar d^2+\mathrm{var}(G))-(n-r)r.\end{aligned}$$

Since Corollary 2.2 provides us with $\bar d^2 \le \rho_1$, this establishes the claimed bound. It is easy to see that it is attained by $G = K_n$. Equality can only hold if equality holds in our bound on $\mathrm{tr}(P^2)$, so only if $p_{ij} = \min\{d_i, d_j\}$ for all $i, j \in \{1,\dots,n\}$, $i \ne j$. This is only possible if G is connected. As we also rely on the bound from Corollary 2.2, this shows that only K_n attains the bound. □

In view of Theorem 2.2, another energy bound can be obtained by using the estimate

$$\bar d^2 \le \frac{\big(\rho_1+(n-1)(n-2)\big)^2}{(2n-3)^2}$$

instead of $\bar d^2 \le \rho_1$ within the previous proof. A bound that is independent of ρ_1 can be made precise by analyzing the function

$$f(\rho_1) := \rho_1 + \Big((n-1)\big(n(n-1)(\rho_1+\mathrm{var}(G))-(n-r)r-\rho_1^2\big)\Big)^{1/2}$$

with respect to ρ_1. The resulting maxima, whose exact formulas are anything but appealing, result in improvements over the energy bounds known so far, but exceed what is necessary to resolve Conjecture 3.1. This is demonstrated in Figure 3.1.

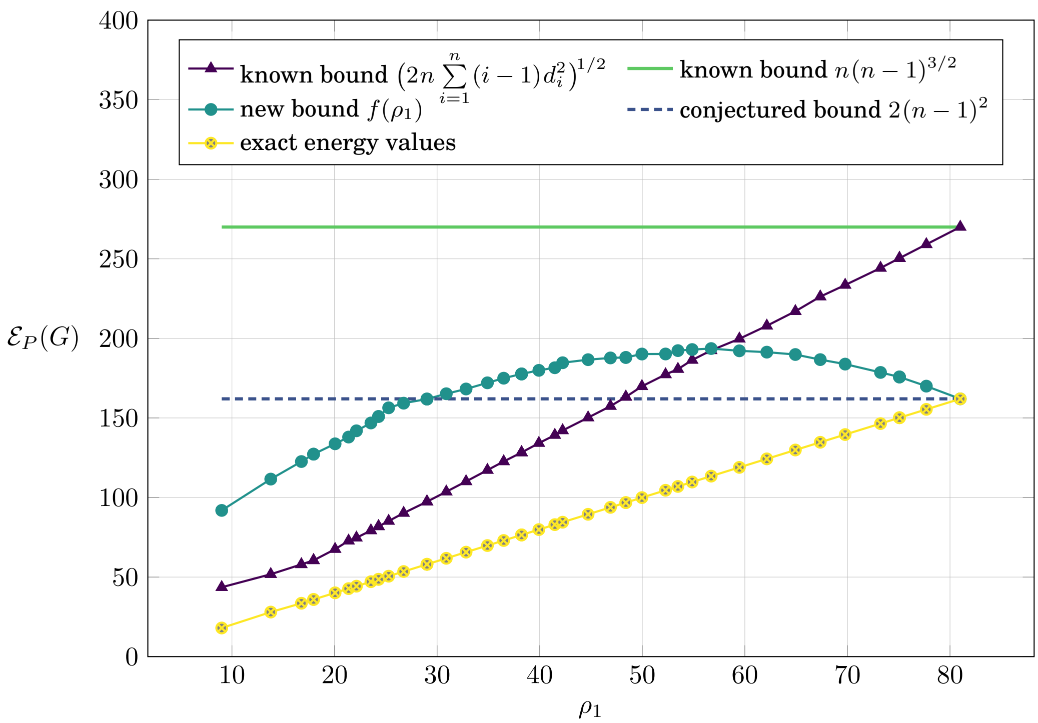

Figure 3.1: Energy bounds and exact values for connected graphs on $n = 10$ vertices with varying edge density.

Whether Conjecture 3.1 holds remains an interesting open question.

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