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Research Article

Polyhedral digital Jordan surfaces

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Abstract

A pair of adjacencies in the digital space $\mathbb{Z}^3$ for every positive integer is introduced. The adjacencies are finer than the 6-adjacency and coarser than the 26-adjacency, and the connectedness provided by them for recognition of digital Jordan surfaces is used. The surfaces are defined to be boundary surfaces of the digital polyhedra that can be face-to-face tiled with certain digital cubes, triangular prisms, and tetrahedra.

Keywords: adjacency; simple graph; connectedness; 3D face-to-face tiling; digital space; digital Jordan surface.

2020 Mathematics Subject Classification: 52C22, 68U05, 51M20.

1. Introduction

Digital Jordan surfaces in the 3D digital space $\mathbb{Z}^3$ approximate the continuous surfaces in the Euclidean space $\mathbb{R}^3$ and, therefore, they should behave analogously. In particular, they are required to satisfy a digital analogue of the 3D Jordan-Brouwer separation theorem (cf. [4]), which means that the surfaces separate $\mathbb{Z}^3$ into two connected components. This is a natural requirement because digital Jordan surfaces represent boundaries of figures in 3D digital images and we want the boundary of a figure to disconnect the interior of the figure from its exterior while both the interior and exterior are connected. Therefore, it is one of the basic problems of digital imagery to find convenient connectedness structures on the digital space $\mathbb{Z}^3$ that permit such digital Jordan surfaces.

In the classical approach to the problem (see [2, 10–12, 15–18], the figures in a 3D digital image consist of digital cubes (voxels) that are connected by 1) faces (6-adjacency) or 2) faces or edges (18-adjacency) or 3) faces or edges or vertices (26-adjacency). Disadvantageously, when using the connectedness given by any one of the three adjacencies, it may happen that the interior of a figure is not connected or is not disconnected from the exterior of the figure. To eliminate this disadvantage, a combination of two kinds of connectedness (adjacencies) has to be used, one for the surface and one for its complement (cf. [3, 10–12]).

The inconvenience of using a combination of two adjacencies in the classical approach is resolved by using the Khalimsky topology proposed by Khalimsky, Kopperman, and Meyer in [7] (see also [9]). The topology provides an adjacency on $\mathbb{Z}^3$ that gives quite a broad variety of digital Jordan surfaces as shown by Kopperman, Meyer, and Wilson in [13] and Melin in [14]. However, it is a disadvantage of the digital Jordan surfaces with respect to the Khalimsky topology that they can never contain a dihedral angle less than $\frac{\pi}{2}$.

In the present paper, we will introduce, for every positive integer n, two new adjacencies in $\mathbb{Z}^3$ having "densities" inversely proportional to n. Both of them are finer than the 6-adjacency and coarser than the 26-adjacency. The connectedness given by the adjacencies will be employed to define digital Jordan surfaces as the boundary surfaces of the digital polyhedra that can be face-to-face tiled with certain digital cubes, triangular prisms, and tetrahedra. The Jordan surfaces obtained may contain acute dihedral angles $\frac{\pi}{4}$ being so more various than the digital Jordan surfaces with respect to the Khalimsky topology. They also generalize the digital Jordan surfaces proposed in [20], which are defined by using a special graph connectedness introduced in [19]. Namely, in [20], the digital Jordan surfaces are defined to be boundary surfaces of the digital polyhedra that can be face-to-face tiled with digital tetrahedra under a restrictive condition imposed on the tiling (at most one of a pair of faces of any of the tetrahedra may be a subset of the surface). The advantage of the digital Jordan surfaces introduced in this note is that they are not restricted by any such a condition. Digital surfaces consisting of faces of certain digital polyhedra are discussed also [1] but our approach is quite different from the one used there.

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The classical connectedness given by a combination of (a pair of) 6-, 18-, and 26-adjacencies is the most frequently used connectedness in graphical software. The mentioned drawback of the connectedness is overcome by the modern efficient hardware. Indeed, nowadays the voxels may be so tiny that the advantage of the digital Jordan surfaces introduced in this note over the other ones is marginal from the viewpoint of applications in digital imagery. Notwithstanding this, the theoretical results presented here may be considered to be a contribution to the development of discrete geometry.

2. Preliminaries

We will use some basic graph-theoretic concepts only and we refer to [6] for them. As usual, by an *adjacency* in a set V we understand a subset $E \subseteq \{\{x,y\};\ x,y \in V,\ x \neq y\}$ (so that an adjacency in a set V is nothing but an irreflexive and symmetric binary relation on V). Elements $x,y \in V$ are said to be E-*adjacent* if $\{x,y\} \in E$ – we then also say that x is an E-*neighbor* of y (and y is an E-*neighbor* of x). By a *graph* we mean an undirected simple graph without loops, hence, a pair $G = (V,E)$ where V is a set whose elements are called the *vertices* of G, and E is an adjacency in V whose elements are called the *edges* of G. A *path* in a graph $G = (V,E)$ is a finite sequence $(x_i|\ i \leq n) = (x_0, x_1, ..., x_n)$ (n a non-negative integer) of pairwise different vertices such that $\{x_i, x_{i+1}\} \in E$ for all $i = 0, 1, ..., n-1$. A subset $A \subseteq V$ is said to be E-*connected* (in G) if, for every pair of vertices $x,y \in A$, there is a path $(x_i|\ i \leq n)$ in G such that $x_0 = x$, $x_n = y$, and $x_i \in A$ for every $i \leq n$. A maximal (with respect to set inclusion) E-connected set is called an E-*component*.

Given graphs $G_1 = (V_1,E_1)$ and $G_2 = (V_2,E_2)$, G_1 is called a *subgraph* of G_2 if $V_1 \subseteq V_2$ and $E_1 \subseteq E_2$. If, moreover, $V_1 = V_2$, then G_1 is said to be a *factor* of G_2. A subgraph (V_1,E_1) of a graph (V_2,E_2) is called an *induced subgraph* of (V_2,E_2) if $E_1 = E_2 \cap \{\{x,y\};\ x,y \in V_1\}$. We speak briefly about the induced subgraph V_1 of (V_2,E_2) in this case. The graphs G_1 and G_2 are said to be *isomorphic* if there is an *isomorphism* between them, i.e., a bijection $f: V_1 \to V_2$ such that $\{x,y\} \in E_1$ if and only if $\{f(x), f(y)\} \in E_2$. Given a set V and an adjacency E in V, subsets $A, B \subseteq V$ are said to be E-*isomorphic* if the induced subgraphs A, B of (V,E) are isomorphic.

We will use the concept of a face-to-face *tiling* (i.e., *tessellation*) with digital polyhedra in $\mathbb{Z}^3$. Recall [5] that such a tiling in $\mathbb{R}^3$ means that any pair of different polyhedra in this tiling is disjoint or meets in one vertex, one full edge or one full face. We will naturally transfer the concept of a face-to-face tiling from $\mathbb{R}^3$ to $\mathbb{Z}^3$. It is well known that a cube in $\mathbb{R}^3$ may be tessellated with six tetrahedra inscribed to the cube (i.e., having the property that each of their vertices is a vertex of the cube) that are congruent to each other, that is, identical up to translation, rotation, or reflection. Such a tessellation of digital cubes will be considered in $\mathbb{Z}^3$.

In this note, instead of voxels used in 3D imagery for partitioning the real space $\mathbb{R}^3$ and creating digital images, we use the points in $\mathbb{Z}^3$, which may be considered to be the center points of the voxels with the edge length 1.

We will transfer some geometric concepts from subsets of the Euclidean space $\mathbb{R}^3$ to subsets of $\mathbb{Z}^3$. More precisely, we will use the concepts of a digital polyhedron, digital cube, digital prism, digital tetrahedron, digital face, etc. to call the subsets of $\mathbb{Z}^3$ that are obtained by the intersections of the corresponding subsets of $\mathbb{R}^3$ (hence, a polyhedron which is understood to be simple, i.e., without any type of holes, cube, prism, tetrahedron, face, etc.) with $\mathbb{Z}^3$. Thus, we will utilize a digitization analogous to the 3D Gauss digitization defined in [8], which uses voxels to digitize subsets of $\mathbb{R}^3$ while we employ the center points of the voxels only. Nevertheless, in the figures presented, digital polyhedra will be demonstrated with their edges represented by line segments rather than digital line segments. If it is clear that a subset of $\mathbb{Z}^3$ and not a continuous subset of $\mathbb{R}^3$ is considered, the adjective “digital” will sometimes be omitted.

In accordance with [20], by an *n-fundamental cube* (n a positive integer) we understand every digital cube $\{(x,y,z) \in \mathbb{Z}^3;\ 2kn \leq x \leq 2kn+2n,\ 2ln \leq y \leq 2ln+2n,\ 2mn \leq z \leq 2mn+2n\}$, $k,l,m \in \mathbb{Z}$. Given an n-fundamental cube C, by a *diagonal rectangle* of C we mean any digital rectangle that is the intersection of C with a plane perpendicular to a face of C and containing one of the two (digital) diagonals of the face. Thus, two parallel sides of a diagonal rectangle of C are diagonals of parallel faces of C and the other two parallel sides are edges of C. Clearly, a diagonal rectangle of an n-fundamental cube separates the cube into two digital right triangular prisms (intersecting in the rectangle, which is a common face of them).

We denote by A_6, A_{18}, and A_{26} the well known 6-, 18-, and 26-*adjacencies* in $\mathbb{Z}^3$. Recall that the adjacencies are defined as follows:

$$A_6 = \left\{\{(x_1,x_2,x_3),(y_1,y_2,y_3)\};\ (x_1,x_2,x_3),(y_1,y_2,y_3) \in \mathbb{Z}^3,\ \sum_{i=1}^{3}|x_i - y_i| = 1\right\},$$

$$A_{18} = \left\{\{(x_1,x_2,x_3),(y_1,y_2,y_3)\};\ (x_1,x_2,x_3),(y_1,y_2,y_3) \in \mathbb{Z}^3,\ \sum_{i=1}^{3}|x_i - y_i| \leq 2,\ \max\{|x_i - y_i|;\ i = 1,2,3\} = 1\right\},$$

$$A_{26} = \{\{(x_1,x_2,x_3),(y_1,y_2,y_3)\};\ (x_1,x_2,x_3),(y_1,y_2,y_3) \in \mathbb{Z}^3,\ \max\{|x_i - y_i|;\ i = 1,2,3\} = 1\}.$$

Clearly, we have $A_6 \subseteq A_{18} \subseteq A_{26}$. Every adjacency A in $\mathbb{Z}^3$ such that $A_6 \subseteq A \subseteq A_{26}$ will be called a *digital adjacency*. For every positive integer n, we define two new digital adjacencies B_n and C_n as follows:

$$\begin{aligned} B_n = A_6 \cup \{\{p_1,p_2\};\ & p_j = (2kn+i_j, 2ln+i_j, m) \text{ for } j \in \{1,2\} \text{ or} \\ & p_j = (2kn+i_j, 2ln-i_j, m) \text{ for } j \in \{1,2\} \text{ or} \\ & p_j = (2kn+i_j, l, 2mn+i_j) \text{ for } j \in \{1,2\} \text{ or} \\ & p_j = (2kn+i_j, l, 2mn-i_j) \text{ for } j \in \{1,2\} \text{ or} \\ & p_j = (k, 2ln+i_j, 2mn+i_j) \text{ for } j \in \{1,2\} \text{ or} \\ & p_j = (k, 2ln+i_j, 2mn-i_j) \text{ for } j \in \{1,2\} \\ \text{where } k,l,m,i_1,i_2 \in \mathbb{Z},\ & |i_1-i_2| = 1\}. \end{aligned}$$

$$\begin{aligned} C_n = B_n \cup \{\{p_1,p_2\};\ & p_j = (2kn+i_j, 2ln+i_j, 2mn+i_j) \text{ for } j \in \{1,2\} \text{ or} \\ & p_j = (2kn+i_j, 2ln+i_j, 2mn-i_j) \text{ for } j \in \{1,2\} \text{ or} \\ & p_j = (2kn-i_j, 2ln+i_j, 2mn+i_j) \text{ for } j \in \{1,2\} \text{ or} \\ & p_j = (2kn-i_j, 2ln+i_j, 2mn-i_j) \text{ for } j \in \{1,2\} \text{ or} \\ \text{where } k,l,m,i_1,i_2 \in \mathbb{Z},\ & |i_1-i_2| = 1\}. \end{aligned}$$

Observe that the graph $(\mathbb{Z}^3, B_n)$ is a factor of $(\mathbb{Z}^3, A_{18})$ but $(\mathbb{Z}^3, C_n)$ is not. It may easily be seen that the graph $(\mathbb{Z}^3, B_n)$ is obtained from $(\mathbb{Z}^3, A_6)$ by adding the edges $\{p,q\} \in A_{18}$ such that both points p and q lie in a diagonal rectangle of an n-fundamental cube. And the graph $(\mathbb{Z}^3, C_n)$ is obtained from $(\mathbb{Z}^3, B_n)$ by adding the edges $\{p,q\} \in A_{26}$ such that both points p and q lie in a body diagonal of an n-fundamental cube. For example, for every positive integer n, B_n-neighbors of $(0,0,0)$ are just the 18-adjacent points to $(0,0,0)$, C_n-neighbors of $(0,0,0)$ are just the 26-adjacent points to $(0,0,0)$, and both B_n-neighbors and C_n-neighbors of $(1,0,0)$ are just the 6-adjacent points to $(1,0,0)$ and the points $(1,j,k) \in \mathbb{Z}^3$ where $|j| = |k| = 1$. Similarly, B_1-neighbors of $(2,0,0)$ are just the 18-adjacent points to $(2,0,0)$, C_1-neighbors of $(2,0,0)$ are just the 26-adjacent points to $(2,0,0)$, and both B_2-neighbors and C_2-neighbors of $(2,0,0)$ are just the 6-adjacent points to $(2,0,0)$ and the points $(2,j,k) \in \mathbb{Z}^3$ where $|j| = |k| = 1$. The connectedness with respect to the Khalimsky topology coincides with the connectedness given by a digital adjacency, the so-called *Khalimsky adjacency*. The Khalimsky adjacency is nothing but the C_1-adjacency (cf. [20]). Since the digital Jordan surfaces with respect to the Khalimsky topology (adjacency) are discussed in [13], n will denote an integer with $n > 1$ in the sequel.

3. Digital Jordan surfaces based on A_6-, B_n-, and C_n-adjacencies

The following statement is obvious:

Proposition 3.1. *Every n-fundamental cube C is A_6-connected and so is every face of C and every set obtained from C by removing some of its faces.*

Definition 3.1. Given an n-fundamental cube C, each of the two digital right triangular prisms such that one of its lateral faces is a diagonal rectangle of C and the other two are faces of C is called an *n-fundamental prism*. The face of an n-fundamental prism that is a diagonal rectangle of C is called the *main face* of the prism.

Thus, every n-fundamental cube may be tessellated with a pair of n-fundamental prisms in six different ways – one of them is shown in Figure 3.1. Consequently, every n-fundamental cube includes 12 n-fundamental prisms, which can be given as sets of points defined by formulas. For example, the two n-fundamental prisms shown in Figure 3.1 can be given (for $k,l,m \in \mathbb{Z}$) as follows:

(1) $\{(x,y,z) \in \mathbb{Z}^3;\ 2kn \le x \le 2kn+2n,\ 2ln \le y \le 2kn+2ln+2n-x,\ 2mn \le z \le 2mn+2n\}$,

(2) $\{(x,y,z) \in \mathbb{Z}^3;\ 2kn \le x \le 2kn+2n,\ 2kn+2ln+2n-x \le y \le 2ln+2n,\ 2mn \le z \le 2mn+2n\}$.

It is left to the reader to give the other ten n-fundamental prisms by formulas.

Proposition 3.2. *Every n-fundamental prism P is B_n-connected and so is every face of P and every set obtained from P by removing some of its faces.*

Proof. Let P be an n-fundamental prism. It may easily be seen that P is A_6-connected and so is every square face of P, every triangular face of P, and every set obtained from P by removing some of its faces. Since $A_6 \subseteq B_n$, the same is true when considering the B_n-connectedness instead of the A_6-connectedness. Since the graph $(\mathbb{Z}^3, B_n)$ is obtained from $(\mathbb{Z}^3, A_6)$ by adding the edges $\{p,q\} \in A_{18}$ such that both points p and q are contained in a diagonal rectangle of an n-fundamental cube, it follows that the main face of P (which is a diagonal rectangle of an n-fundamental cube C) is B_n-isomorphic to each of the two square faces of P (which are faces of C). Hence, the main face of P is B_n-connected. □

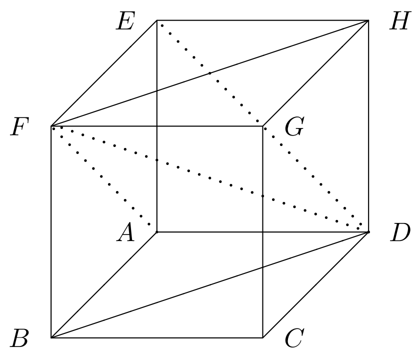

Figure 3.1: An n-fundamental cube $ABCDHEFG$, its tessellation with two n-fundamental prisms, and a tessellation of the n-fundamental prism $ABDHEF$ with three n-fundamental tetrahedra where $A = (2kn, 2ln, 2mn)$, $B = (2kn+2n, 2ln, 2mn)$, $C = (2kn+2n, 2ln+2n, 2mn)$, $D = (2kn, 2ln+2n, 2mn)$, $E = (2kn, 2ln, 2mn+2n)$, $F = (2kn+2n, 2ln, 2mn+2n)$, $G = (2kn + 2n, 2ln + 2n, 2mn + 2n)$, $H = (2kn, 2ln + 2n, 2mn + 2n)$, $k, l, m \in \mathbb{Z}$.

Definition 3.2. By an n-*fundamental tetrahedron* we understand any of the three digital tetrahedra obtained by dividing an n-fundamental prism by the planes ADF and DEF where A, D, E, F are vertices of the prism such that the line segment DF is a diagonal of the main face, the line segment AF is a diagonal of a square face, and the line segment DE is a diagonal of the other square face of the prism.

Clearly, every n-fundamental prism may be tessellated with three n-fundamental tetrahedra in two different ways (one of them is demonstrated in Figure 3.1 by dotted line segments). Hence, every n-fundamental cube may be tessellated with six n-fundamental tetrahedra in twelve different ways. It may easily be seen that, for every n-fundamental cube, there are 24 n-fundamental tetrahedra included in the cube. Evidently, all n-fundamental tetrahedra are congruent to each other. Of course, the n-fundamental tetrahedra can be given as sets of points defined by formulas. For example, the three n-fundamental tetrahedra shown in Figure 3.1 can be given (for $k, l, m \in \mathbb{Z}$) as follows:

(1) $\{(x, y, z) \in \mathbb{Z}^2;\ 2kn \le x \le 2kn + 2n,\ 2ln \le y \le 2kn + 2ln + 2n - x,\ 2mn \le z \le x + 2mn - 2kn\}$,

(2) $\{(x, y, z) \in \mathbb{Z}^2;\ 2kn \le x \le 2kn + 2n,\ 2ln \le y \le 2kn + 2ln + 2n - x,\ 2ln + 2mn + 2n - y \le z \le 2kn + 2n\}$,

(3) $\{(x, y, z) \in \mathbb{Z}^2;\ 2kn \le x \le 2kn + 2n,\ 2ln \le y \le 2kn + 2ln + 2n - x,\ x + 2mn - 2kn \le z \le 2ln + 2mn + 2n - y\}$.

It is left to the reader to give the other 21 n-fundamental tetrahedra by formulas.

Proposition 3.3. *Every n-fundamental tetrahedron T is C_n-connected and so is every face of T and every set obtained from T by removing some of its faces.*

Proof. Let T be an n-fundamental tetrahedron. It may easily be seen that T is A_6-connected and so is every isosceles (right) triangular face of T and every set obtained from T by removing some of its faces. As $A_6 \subseteq C_n$, the same is true when considering the C_n-connectedness instead of the A_6-connectedness. Since the graph $(\mathbb{Z}^3, C_n)$ is obtained from $(\mathbb{Z}^3, B_n)$ by adding the edges $\{p, q\} \in A_{26}$ such that both points p and q are contained in a body diagonal of an n-fundamental cube, each of the two non-isosceles (right) triangular faces of T is C_n-isomorphic to each of the two isosceles triangular faces of T. Hence, each of the two faces is C_n-connected. □

Definition 3.3. Let A be a digital adjacency. An A-connected subset $S \subseteq \mathbb{Z}^3$ is called an A-*Jordan surface* if the subgraph $\mathbb{Z}^3 - S$ of $(\mathbb{Z}^3, A)$ has exactly two A-components V_1 and V_2, one finite and the other infinite, and the sets $V_1 \cup S$ and $V_2 \cup S$ are A-connected.

Theorem 3.1 (Digital Jordan Surface Theorem I). *Let $S \subseteq \mathbb{Z}^3$ be a polyhedral surface and V be the polyhedron that is bounded by S. If V can be face-to-face tiled with a set of n-fundamental cubes, prisms or tetrahedra, then S is an A_6-, B_n- or C_n-Jordan surface, respectively.*

Proof. Let S satisfy the conditions of the statement and let V can be face-to-face tiled with a set $\mathcal{P}$ of n-fundamental cubes. Then S is the union of some (finitely many) faces of certain n-fundamental cubes belonging to $\mathcal{P}$. The faces are A_6-connected by Proposition 3.1 and can be ordered into a (finite) sequence such that each of its terms meets the union of its predecessors. Therefore, S is A_6-connected.

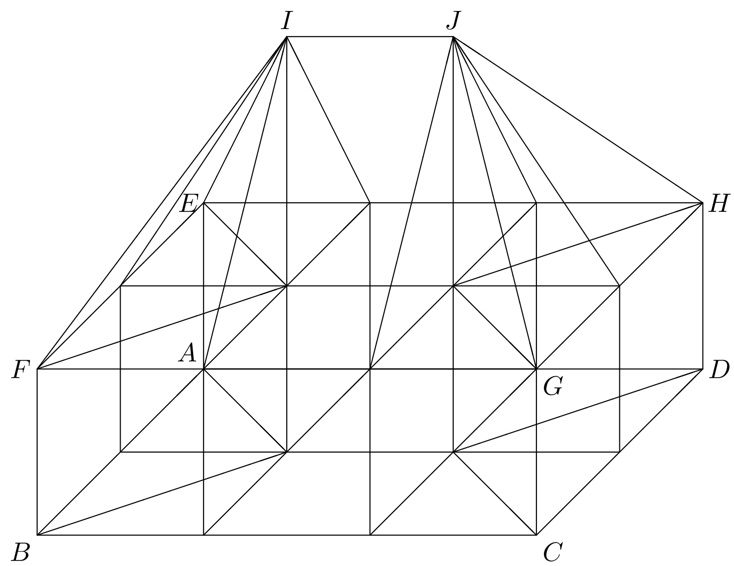

Figure 3.2: A nonahedron $ABCDEFGHIJ$ in $\mathbb{Z}^3$ tiled with twenty n-fundamental polyhedra where $A = (2kn, 2ln, 2mn)$, $B = (2kn + 4n, 2ln, 2mn)$, $C = (2kn + 4n, 2ln + 6n, 2mn)$, $D = (2kn, 2ln + 6n, 2mn)$, $E = (2kn, 2ln, 2mn + 2n)$, $F = (2kn + 4n, 2ln, 2mn + 2n)$, $G = (2kn + 4n, 2ln + 6n, 2mn + 2n)$, $H = (2kn, 2ln + 6n, 2mn + 2n)$, $I = (2kn + 2n, 2ln + 2n, 2mn + 4n)$, $J = (2kn + 2n, 2ln + 4n, 2mn + 4n)$,$k, l, m \in \mathbb{Z}$.

Put $V_1 = V - S$ and $V_2 = \mathbb{Z}^3 - V$. Clearly, the elements of $\mathcal{P}$ can also be ordered into a (finite) sequence such that each of its terms meets the union of its predecessors. Since V is the union of all elements of $\mathcal{P}$, which are n-fundamental cubes, $V = V_1 \cup S$ is C_n-connected because every n-fundamental cube is A_6-connected by Proposition 3.1. Similarly, the set $W = (\mathbb{Z}^3 - V) \cup S = V_2 \cup S$ is the union of an (infinite) sequence of n-fundamental cubes such that each term of the sequence meets the union of its predecessors. Hence, W is A_6-connected, too. Analogously, the sets $V_1 = V - S$ and $V_2 = \mathbb{Z}^3 - V$ are the unions of (finite and infinite, respectively) sequences of subsets of $\mathbb{Z}^3$ obtained from n-fundamental cubes by removing some of their faces. These subsets are A_6-connected by Proposition 3.1. In the sequences, each term meets the union of its predecessors, hence V_1 and V_2 are C_n-connected.

It follows from the definition of the adjacency A_6 that $\mathbb{Z}^3 - S$ is not A_6-connected, i.e., that $\{p, q\} \notin A_6$ whenever $p \in V_1$ and $q \in V_2$. Thus, V_1 and V_2 are A_6-components of $\mathbb{Z}^3 - S$, V_1 finite and V_2 infinite. We have proved that S is an A_6-Jordan surface if V can be face-to-face tiled with a set of n-fundamental cubes. For the other two cases, the proofs are analogical: we just replace n-fundamental cubes with n-fundamental prisms (n-fundametal tetrahedra), A_6-connectedness with B_n-connectedness (C_n-connectedness), and apply Proposition 3.2 (Proposition 3.3) instead of Proposition 3.1. □

The n-fundamental cubes, n-fundamental prisms, and n-fundamental tetrahedra will be called the *n-fundamental polyhedra*.

Corollary 3.1 (Digital Jordan Surface Theorem II). *Let $S \subseteq \mathbb{Z}^3$ be a polyhedral surface and V be the polyhedron that is bounded by S. If V can be face-to-face tiled with a set of n-fundamental polyhedra, then S is a C_n-Jordan surface.*

Proof. Let S satisfy the conditions of the statement and let V can be face-to-face tiled with a set $\mathcal{P}$ of n-fundamental polyhedra. Since $A_6 \subseteq C_n$ and $B_n \subseteq C_n$, it follows from Theorem 3.1 that the sets $V - S$ and $\mathbb{Z}^3 - V$ are C_n-connected and so are the sets V and $(\mathbb{Z}^3 - V) \cup S$. It immediately follows from the definition of C_n that $\{p, q\} \notin C_n$ whenever $p \in (V - S)$ and $q \in (\mathbb{Z}^3 - V)$. Therefore, $V - S$ and $\mathbb{Z}^3 - V$ are C_n-components of $\mathbb{Z}^3 - S$. □

Example 3.1. In Figure 3.2, a digital surface in $\mathbb{Z}^3$ is displayed that is a boundary of a digital nonahedron which may be face-to-face tiled with twenty n-fundamental polyhedra (two n-fundamental cubes, ten n-fundamental prisms, and eight n-fundamental tetrahedra) such that the conditions of Corollary 3.1 are satisfied. Thus, the surface is a C_n-Jordan surface.

Example 3.2. In Figure 3.3, a digital surface in $\mathbb{Z}^3$ is displayed that is a boundary of a digital heptahedron which may be face-to-face tiled with eight n-fundamental polyhedra (four n-fundamental prisms and four n-fundamental tetrahedra) such that the conditions of Corollary 3.1 are satisfied. Thus, also this surface is a C_n-Jordan surface.

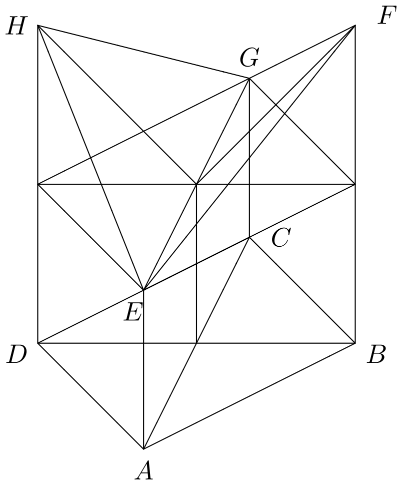

Figure 3.3: A heptahedron $ABCDEFGH$ in $\mathbb{Z}^3$ tiled with eight n-fundamental polyhedra where $A = (2kn+2n, 2ln, 2mn)$, $B = (2kn, 2ln+2n, 2mn)$, $C = (2kn-2n, 2ln, 2mn)$, $D = (2kn, 2ln-2n, 2mn)$, $E = (2kn+2n, 2ln, 2mn+2n)$, $F = (2kn, 2ln+2n, 2mn+4n)$, $G = (2kn-2n, 2ln, 2mn+2n)$, $H = (2kn, 2ln-2n, 2mn+4n)$, $k, l, m \in \mathbb{Z}$.

4. Conclusion

We have introduced, for every positive integer n, a pair of adjacencies B_n, C_n, $B_n \subseteq C_n$, in the digital space $\mathbb{Z}^3$, which are then used to recognize digital Jordan surfaces. The surfaces are defined to be boundary surfaces of the digital polyhedra that can be face-to-face tiled with (digital) cubes, triangular prisms, and tetrahedra. The adjacency C_n constitures a rich variety of digital Jordan surfaces and its advantage over the classical adjacencies A_6, A_{12}, and A_{26} is that it need not be combined with any other adjacency to provide a convenient connectedness in $\mathbb{Z}^3$. The surfaces may contain acute dihedral angles $\frac{\pi}{4}$, which is an advantage over the Jordan surfaces with respect to the Khalimsky topology. The variety of digital Jordan surfaces constituted by C_n is also wider than the one obtained in [20] by using the graph connectedness introduced in [19]. Namely, in [20], a digital Jordan surface is defined to be the boundary of a digital polyhedron that can be face-to-face tiled with digital tetrahedra under the restriction that certain faces of the tetrahedra cannot be included in the boundary.

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