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Research Article

The preservation property of Brouwer's conjecture

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Abstract

Let G be a simple graph with n vertices and $e(G)$ edges. Brouwer's conjecture states that the sum of the k largest Laplacian eigenvalues of G is at most $e(G) + \frac{k^2+k}{2}$ for $k = 1, 2, \ldots, n$. Torres and Trevisan [*Linear Algebra Appl.* **685** (2024) 66–76] showed that if Brouwer's conjecture holds for two simple graphs G_1 and G_2, then it also holds for the Cartesian product of G_1 and G_2. Inspired by this result, we say that an operation on G_1 and G_2 satisfies the preservation property of Brouwer's conjecture when the following statement is true: if Brouwer's conjecture holds for G_1 and G_2, then Brouwer's conjecture also holds for the graph obtained by applying the operation under consideration on G_1 and G_2. In this paper, we study the preservation property of Brouwer's conjecture under some edge addition operations, and hence we extend the results of Wang, Huang, and Liu [*Math. Comput. Model.* **56** (2012) 60–68].

Keywords: graph operation; Brouwer's conjecture; Laplacian eigenvalue.

2020 Mathematics Subject Classification: 05C50.

1. Introduction

Let G be a simple graph with the vertex set $V(G)$ and edge set $E(G)$. The number of edges of G is denoted by $e(G)$. The Laplacian matrix of G, denoted by $L(G)$, is given by $L(G) = D(G) - A(G)$, where $D(G)$ is the diagonal matrix of vertex degrees and $A(G)$ is the adjacency matrix. Clearly, $L(G)$ is symmetric and positive semidefinite, and its eigenvalues can be ordered as

$$\mu_1(G) \geq \mu_2(G) \geq \cdots \mu_{n-1}(G) \geq \mu_n(G) = 0.$$

The set of Laplacian eigenvalues is called the Laplacian spectrum of G.

The sum of the k largest Laplacian eigenvalues of a graph G is defined as

$$S_k(G) = \sum_{i=1}^{k} \mu_i(G),$$

where $k = 1, 2, \ldots, n$. Motivated by the Grone-Merris-Bai Theorem [1,17], Brouwer [3] proposed the following conjecture:

Conjecture 1.1. (Brouwer's conjecture) *For any graph G with n vertices and for any $k \in \{1, 2, \cdots, n\}$,*

$$S_k(G) \leq e(G) + \frac{k^2+k}{2}.$$

This conjecture is directly related to the distribution of Laplacian eigenvalues of graphs, which is a fundamental problem in spectral graph theory. Until now, Conjecture 1.1 has been verified for all graphs with at most 11 vertices [7], all graphs when $k = 2, n-2, n-3$ [5,19], trees [19], threshold graphs [19], unicyclic graphs [8,29], bicyclic graphs [8], tricyclic graphs when $k \neq 3$ [21,29], regular graphs [2,24], split graphs [2,4,24], cographs [2,24], planar graphs when $k \geq 11$, and bipartite graphs when $k \geq \sqrt{32n}$ [7]. In [5], Chen showed that if Brouwer's conjecture is true for all graphs when $k = p$ $(1 \leq p \leq (n-1)/2)$, then it is also true for all graphs when $k = n - p - 1$. This means that it suffices to prove Conjecture 1.1 for all graphs when $1 \leq k \leq (n-1)/2$. Moreover, Wang et al. [29] showed that Brouwer's conjecture holds for all graphs if and only if Brouwer's conjecture holds for all connected graphs. Hence, it is sufficient to consider only connected graphs. Recently, Li and Guo [22] proposed the full Brouwer's conjecture and proved the conjecture holds for $k = 2$, which also confirms the conjecture of Guan et al. [18]. There are also some other classes of graphs for which various parameters meet certain conditions; for instance, we refer the reader to [6,12–16,25,27].

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On the other hand, the Laplacian energy of a graph G with n vertices is defined as

$$LE(G) = \sum_{i=1}^{n} \left| \mu_i(G) - \frac{2e(G)}{n} \right|,$$

which is an important parameter in the molecular orbital theory of conjugated molecules [26]. Since

$$LE(G) = \max_{1 \le k \le n} \left\{ 2S_k(G) - \frac{4ke(G)}{n} \right\},$$

it follows that every upper bound on $S_k(G)$ yields a corresponding upper bound on $LE(G)$. In particular, Fritscher et al. [10,11] studied the extremal value of the Laplacian energy of trees based on the upper bound of the sum of the k largest Laplacian eigenvalues. Therefore, the solution to the Brouwer's conjecture directly promotes the study of the Laplacian energy of a graph.

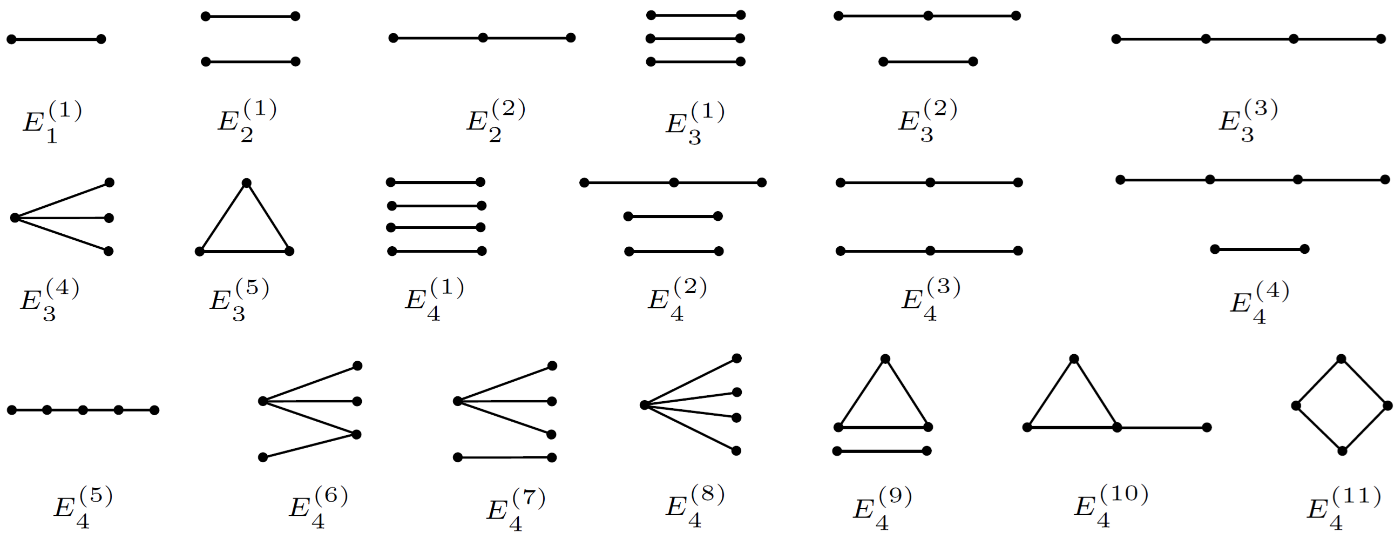

Figure 1.1: The graphs $E_1^{(1)}$, $E_2^{(t)}$ $(t = 1, 2)$, $E_3^{(j)}$ $(j = 1, 2, \dots, 5)$, and $E_4^{(l)}$ $(l = 1, 2, \dots, 11)$.

Let G_1 and G_2 be two simple graphs. Let $G_1 \diamond G_2$ denote the graph obtained from G_1 and G_2 by connecting a vertex of G_1 with $p \ge 1$ vertices of G_2. The graphs $E_1^{(1)}$, $E_2^{(t)}$ $(t = 1, 2)$, $E_3^{(j)}$ $(j = 1, 2, \dots, 5)$, and $E_4^{(l)}$ $(l = 1, 2, \dots, 11)$, are shown in Figure 1.1. Let $G_1E_1^{(1)}G_2$, $G_1E_2^{(t)}G_2$, $G_1E_3^{(j)}G_2$, and $G_1E_4^{(l)}G_2$, respectively, denote the graph obtained from G_1 and G_2 by inserting a graph $E_1^{(1)}$, $E_2^{(t)}$, $E_3^{(j)}$, or $E_4^{(l)}$ between $V(G_1)$ and $V(G_2)$ for $t = 1, 2$, $j = 1, 2, \dots, 5$ and $l = 1, 2, \dots, 11$. Note that the number of newly added vertices in $G_1E_1^{(1)}G_2$, $G_1E_2^{(t)}G_2$, $G_1E_3^{(j)}G_2$, or $G_1E_4^{(l)}G_2$ is at most 3 for $t = 1, 2$, $j = 1, 2, \dots, 5$ and $l = 1, 2, \dots, 11$; see Figure 1.2.

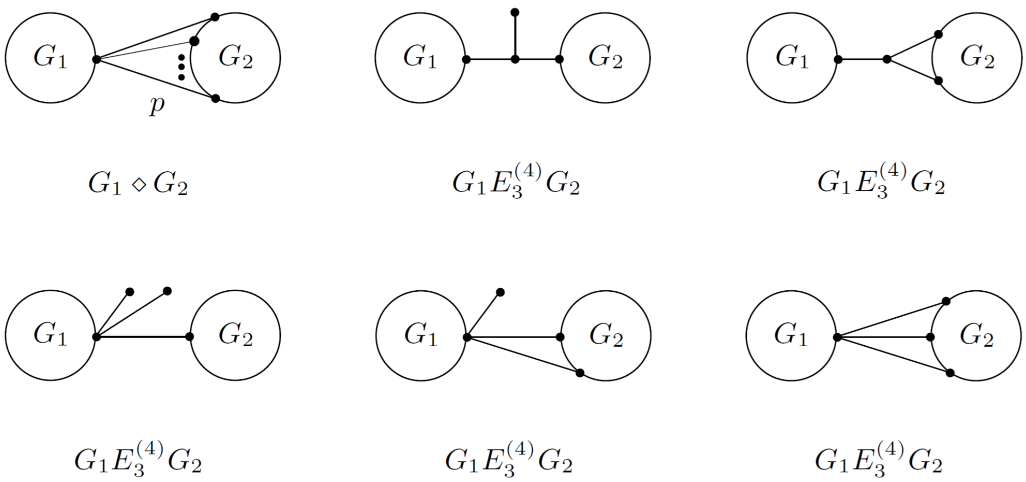

Figure 1.2: The graphs $G_1 \diamond G_2$ and $G_1E_3^{(4)}G_2$.

In 2024, Torres and Trevisan [28] showed that if Brouwer's conjecture holds for G_1 and G_2, then it also holds for the Cartesian product of G_1 and G_2. In this paper, we show that if Brouwer's conjecture holds for G_1 and G_2, then this conjecture is also true for $G_1 \diamond G_2$, $G_1 E_1^{(1)} G_2$, $G_1 E_2^{(t)} G_2$, $G_1 E_3^{(j)} G_2$, and $G_1 E_4^{(l)} G_2$ for $t = 1, 2$, $j = 1, 2, \ldots, 5$ and $l = 1, 2, \ldots, 11$.

2. Preliminaries

Let G_1 and G_2 be two disjoint graphs. Let $G_1 \cup G_2$ denote the disjoint union of G_1 and G_2. If $u \in V(G_1)$ and $v \in V(G_2)$, then the coalescence $G_1 \circ G_2$ of G_1 and G_2 is the graph obtained by identifying the vertices u and v. The coalescence of three or more disjoint graphs is denoted by $G_1 \circ G_2 \circ \cdots \circ G_s$ $(s \geq 3)$. If graphs $G_1, \ldots, G_s$ are isomorphic, then $G_1 \circ G_2 \circ \cdots \circ G_s$ is abbreviated as $s \circ G_1$. In particular, $G_1 * G_2 * \cdots * G_s$ $(s \geq 3)$ denotes the common vertex coalescing. The complete graph with n vertices is denoted by K_n. The four examples of the coalescence of graphs are shown in Figure 2.1.

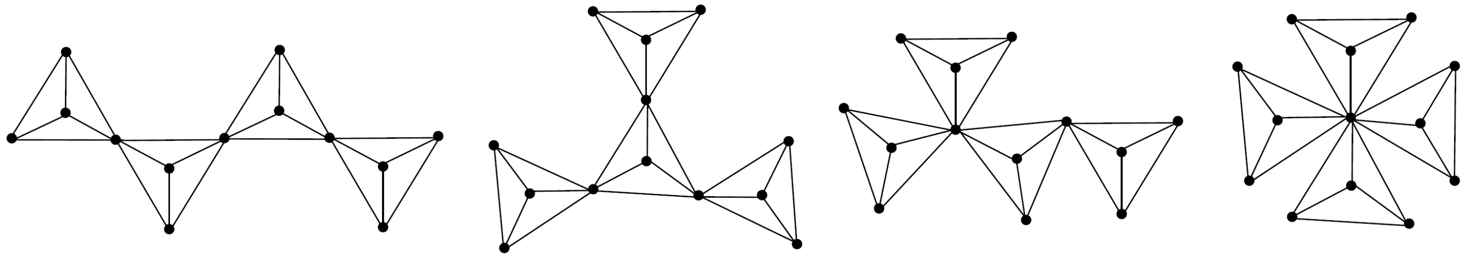

Figure 2.1: Four possibilities of the graph $4 \circ K_4$.

Lemma 2.1 (see [9]). *Let M and N be two real symmetric matrices of order n. Then for $1 \leq k \leq n$, the following inequality holds:*

$$\sum_{i=1}^{k} \lambda_i(M+N) \leq \sum_{i=1}^{k} \lambda_i(M) + \sum_{i=1}^{k} \lambda_i(N).$$

Lemma 2.2 (see [20]). *Let G be a graph with n vertices. Then $\mu_1(G) \leq n$.*

Lemma 2.3 (see [23,30]). *Let T_n and U_n be a tree and a unicyclic graph on n vertices, respectively. Also, consider the graphs $T_n^1, T_n^2, \ldots, T_n^5, U_n^1, U_n^2, U_n^3$, and U_n^4, shown in Figure 2.2.*

(i). *If $T_n \notin \{T_n^1, T_n^2\}$, then $\mu_1(T_n) < n - 1$.*

(ii). *If $T_n \notin \{T_n^1, T_n^2, T_n^3, T_n^4, T_n^5\}$, then $\mu_1(T_n) < n - 2$.*

(iii). *If $U_n \notin \{U_n^1, U_n^2, U_n^3, U_n^4\}$, then $\mu_1(U_n) < n - 1$.*

By using Lemmas 2.2 and 2.3, and simple calculation, we obtain the next result.

Lemma 2.4. *Let G be a connected graph with n vertices and $e(G)$ edges.*

(i). *If $G \notin \{T_n^1, T_n^2\}$, then $\mu_1(G) \leq e(G)$.*

(ii). *If $G \notin \{T_n^1, T_n^2, U_n^1\}$, then $\mu_1(G) \leq e(G) - 0.7$.*

(iii). *If $G \notin \{T_n^1, T_n^2, T_n^3, T_n^4, T_n^5, U_n^1, U_n^2, U_n^3, U_n^4\}$, then $\mu_1(G) \leq e(G) - 1$.*

3. Main results

Theorem 3.1. *Let p be a positive integer. Let G_1 and G_2 be two connected graphs with n_1 and n_2 vertices, respectively. If $G_i \notin \{T_{n_i}^1, T_{n_i}^2\}$ is any of the graphs shown in Figure 2.2, $e(G_i) \geq p$, and*

$$S_{k_i}(G_i) \leq e(G_i) + \frac{k_i^2 + k_i}{2}$$

for $k_i = 1, 2, \ldots, n_i$ and $i = 1, 2$, then for $1 \leq k \leq n_1 + n_2$, the following inequality holds:

$$S_k(G_1 \diamond G_2) \leq e(G_1 \diamond G_2) + \frac{k^2 + k}{2}.$$

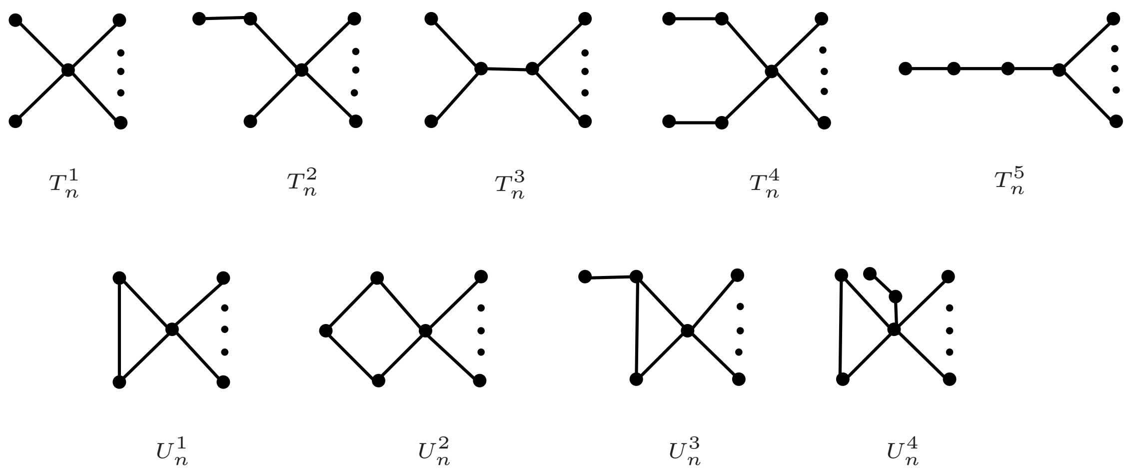

Figure 2.2: The graphs $T_n^1, T_n^2, \ldots, T_n^5, U_n^1, U_n^2, U_n^3$, and U_n^4.

Proof. Assume that k_i of the k largest Laplacian eigenvalues of $G_1 \cup G_2$ come from the Laplacian spectrum of G_i, where $i = 1, 2$ and $k_1 + k_2 = k$. If $k_1 k_2 = 0$, then we suppose that $k_2 = 0$, without loss of generality. Then $k_1 = k$ and by Lemma 2.1, we have

$$\begin{aligned} S_k(G_1 \diamond G_2) &\leq S_k(G_1 \cup G_2) + S_k(K_{1,p}) \\ &= S_k(G_1) + p + 1 + (k-1) \\ &\leq e(G_1) + \frac{k^2+k}{2} + p + p \\ &= e(G_1) + e(G_2) + p + \frac{k^2+k}{2} + p - e(G_2) \\ &\leq e(G_1 \diamond G_2) + \frac{k^2+k}{2}. \end{aligned}$$

If $k_1 k_2 \geq k_1 + k_2$, then by Lemma 2.1, we have

$$\begin{aligned} S_k(G_1 \diamond G_2) &\leq S_k(G_1 \cup G_2) + S_k(K_{1,p}) \\ &\leq e(G_1) + \frac{k_1^2+k_1}{2} + e(G_2) + \frac{k_2^2+k_2}{2} + p + k \\ &= e(G_1) + e(G_2) + p + \frac{k_1^2+k_2^2+k_1+k_2}{2} + k_1 + k_2 \\ &\leq e(G_1 \diamond G_2) + \frac{k_1^2+k_2^2+k_1+k_2}{2} + k_1 k_2 \\ &= e(G_1 \diamond G_2) + \frac{k^2+k}{2}. \end{aligned}$$

If $k_1 k_2 < k_1 + k_2$, without loss of generality, suppose that $k_1 = 1$ and $k_2 = k-1$. By Lemmas 2.1 and 2.4, we have

$$\begin{aligned} S_k(G_1 \diamond G_2) &\leq S_k(G_1 \cup G_2) + S_k(K_{1,p}) \\ &\leq \mu_1(G_1) + e(G_2) + \frac{(k-1)^2+k-1}{2} + p + k \\ &= e(G_1) + e(G_2) + p + \frac{(k-1)^2+3k-1}{2} + \mu_1(G_1) - e(G_1) \\ &\leq e(G_1 \diamond G_2) + \frac{k^2+k}{2}. \end{aligned}$$

This completes the proof. □

Corollary 3.1. *If $n_i \geq 4$ for $i = 1, 2, \ldots, q$, with $q \geq 2$, then Brouwer's conjecture holds for $K_{n_1} \circ K_{n_2} \circ \cdots \circ K_{n_q}$.*

Proof. Without loss of generality, we assume that $n_2 \geq n_1 \geq 4$. Clearly, Brouwer's conjecture holds for K_{n_1} and K_{n_2}. Note that $K_{n_1} \circ K_{n_2}$ can be regarded as $K_{n_1} \diamond K_{n_2-1}$ with $p = n_2 - 1 \leq e(K_{n_2})$. By Theorem 3.1, Brouwer's conjecture holds for $K_{n_1} \diamond K_{n_2-1}$. Applying mathematical induction on q, we have the proof of the corollary. □

Chen [5] showed that Conjecture 1.1 is true for all graphs of order n with $k = n - 3$. Since the number of newly added vertices in $G_1E_1^{(1)}G_2$, $G_1E_2^{(t)}G_2$, $G_1E_3^{(j)}G_2$, or $G_1E_4^{(l)}G_2$ is at most 3 for $t = 1, 2$, $j = 1, 2, \ldots, 5$, and $l = 1, 2, \ldots, 11$, the condition $1 \leq k \leq n_1 + n_2$ is sufficient in the following discussion.

Theorem 3.2. *Let G_1 and G_2 be two connected graphs with n_1 and n_2 vertices, respectively. Also, let*

$$S_{k_i}(G_i) \leq e(G_i) + \frac{k_i^2 + k_i}{2}$$

for $k_i = 1, 2, \ldots, n_i$ and $i = 1, 2$. Consider the graphs $T_n^1, T_n^2, \ldots, T_n^5, U_n^1, U_n^2, U_n^3$, and U_n^4 as shown in Figure 2.2.

(i). *If $e(G_i) \geq 2$, $i = 1, 2$, then for $1 \leq k \leq n_1 + n_2$, $s = 1, 2$, $t = 1, 2$ and $s \geq t$,*

$$S_k(G_1E_s^{(t)}G_2) \leq e(G_1E_s^{(t)}G_2) + \frac{k^2 + k}{2}.$$

(ii). *If $e(G_i) \geq 3$, $G_i \notin \{T_{n_i}^1, T_{n_i}^2\}$ for $i = 1, 2$, then for $1 \leq k \leq n_1 + n_2$ and $j = 1, 2, \ldots, 5$,*

$$S_k(G_1E_3^{(j)}G_2) \leq e(G_1E_3^{(j)}G_2) + \frac{k^2 + k}{2}.$$

(iii). *If $e(G_i) \geq 4$, $G_i \notin \{T_{n_i}^1, T_{n_i}^2\}$ for $i = 1, 2$, then for $1 \leq k \leq n_1 + n_2$ and $l = 1, 2, 3, 7, 8$,*

$$S_k(G_1E_4^{(l)}G_2) \leq e(G_1E_4^{(l)}G_2) + \frac{k^2 + k}{2}.$$

If $e(G_i) \geq 4$, $G_i \notin \{T_{n_i}^1, T_{n_i}^2, U_{n_i}^1\}$ for $i = 1, 2$, then for $1 \leq k \leq n_1 + n_2$ and $l = 4, 5, 6$,

$$S_k(G_1E_4^{(l)}G_2) \leq e(G_1E_4^{(l)}G_2) + \frac{k^2 + k}{2}.$$

If $e(G_i) \geq 4$, $G_i \notin \{T_{n_i}^1, \ldots, T_{n_i}^5, U_{n_i}^1, \ldots U_{n_i}^4\}$ for $i = 1, 2$, then for $1 \leq k \leq n_1 + n_2$ and $l = 9, 10, 11$,

$$S_k(G_1E_4^{(l)}G_2) \leq e(G_1E_4^{(l)}G_2) + \frac{k^2 + k}{2}.$$

Proof. Conjecture 1.1 is true for all graphs when $k = 2$ [19]. Now, we consider the case where $k \geq 3$. Assume that k_i of the k largest Laplacian eigenvalues of $G_1 \cup G_2$ come from the Laplacian spectrum of G_i, where $i = 1, 2$ and $k_1 + k_2 = k$.

(i). The desired result holds for the graphs $G_1E_1^{(1)}G_2$ and $G_1E_2^{(1)}G_2$ due to [29]. Now, we only need to prove the result for the graph $G_1E_2^{(2)}G_2$. If $k_1k_2 = 0$, without loss of generality, suppose that $k_2 = 0$. Then $k_1 = k$ and by Lemma 2.1, we have

$$\begin{aligned} S_k(G_1E_2^{(2)}G_2) &\leq S_k(G_1 \cup G_2) + S_k(E_2^{(2)}) = S_k(G_1) + 4 \\ &\leq e(G_1) + \frac{k^2 + k}{2} + 4 = e(G_1) + e(G_2) + 2 + \frac{k^2 + k}{2} + 2 - e(G_2) \\ &\leq e(G_1E_2^{(2)}G_2) + \frac{k^2 + k}{2}. \end{aligned}$$

If $k_1k_2 \geq 2$, then by Lemma 2.1, we have

$$\begin{aligned} S_k(G_1E_2^{(2)}G_2) &\leq S_k(G_1 \cup G_2) + S_k(E_2^{(2)}) \\ &\leq e(G_1) + \frac{k_1^2 + k_1}{2} + e(G_2) + \frac{k_2^2 + k_2}{2} + 4 = e(G_1) + e(G_2) + 2 + \frac{k_1^2 + k_2^2 + k_1 + k_2}{2} + 2 \\ &\leq e(G_1E_2^{(2)}G_2) + \frac{k_1^2 + k_2^2 + k_1 + k_2}{2} + k_1k_2 = e(G_1E_2^{(2)}G_2) + \frac{k^2 + k}{2}. \end{aligned}$$

(ii). If $k_1k_2=0$, without loss of generality, suppose that $k_2=0$. Then $k_1=k$ and by Lemma 2.1, for $E_3^{(j)}$, $j=1,2,\ldots,5$, we have

$$\begin{aligned} S_k(G_1E_3^{(j)}G_2) &\leq S_k(G_1\cup G_2)+S_k(E_3^{(j)})=S_k(G_1)+6 \\ &\leq e(G_1)+\frac{k^2+k}{2}+6=e(G_1)+e(G_2)+3+\frac{k^2+k}{2}+3-e(G_2) \\ &\leq e(G_1E_3^{(j)}G_2)+\frac{k^2+k}{2}. \end{aligned}$$

If $k_1k_2\geq 3$, then by Lemma 2.1, we have

$$\begin{aligned} S_k(G_1E_3^{(j)}G_2) &\leq S_k(G_1\cup G_2)+S_k(E_3^{(j)}) \\ &\leq e(G_1)+\frac{k_1^2+k_1}{2}+e(G_2)+\frac{k_2^2+k_2}{2}+6=e(G_1)+e(G_2)+3+\frac{k_1^2+k_2^2+k_1+k_2}{2}+3 \\ &\leq e(G_1E_3^{(j)}G_2)+\frac{k_1^2+k_2^2+k_1+k_2}{2}+k_1k_2=e(G_1E_3^{(j)}G_2)+\frac{k^2+k}{2}. \end{aligned}$$

If $k_1k_2=2$, without loss of generality, suppose that $k_1=1$ and $k_2=2$. Then, by Lemmas 2.1 and 2.4, we have

$$\begin{aligned} S_3(G_1E_3^{(j)}G_2) &\leq S_3(G_1\cup G_2)+S_3(E_3^{(j)}) \\ &\leq \mu_1(G_1)+e(G_2)+3+6=e(G_1)+e(G_2)+3+6+\mu_1(G_1)-e(G_1) \\ &\leq e(G_1E_3^{(j)}G_2)+6. \end{aligned}$$

(iii). If $k_1k_2=0$, without loss of generality, suppose that $k_2=0$. Then $k_1=k$ and by Lemma 2.1, for $E_4^{(l)}$, $l=1,2,\ldots,11$, we have

$$\begin{aligned} S_k(G_1E_4^{(l)}G_2) &\leq S_k(G_1\cup G_2)+S_k(E_4^{(l)})=S_k(G_1)+S_k(E_4^{(l)}) \\ &\leq e(G_1)+\frac{k^2+k}{2}+8=e(G_1)+e(G_2)+4+\frac{k^2+k}{2}+4-e(G_2) \\ &\leq e(G_1E_4^{(l)}G_2)+\frac{k^2+k}{2}. \end{aligned}$$

If $k_1k_2\geq 4$, for $E_4^{(l)}$, $l=1,2,\ldots,11$, by Lemma 2.1, we have

$$\begin{aligned} S_k(G_1E_4^{(l)}G_2) &\leq S_k(G_1\cup G_2)+S_k(E_4^{(l)}) \\ &\leq e(G_1)+\frac{k_1^2+k_1}{2}+e(G_2)+\frac{k_2^2+k_2}{2}+S_k(E_4^{(l)}) \\ &\leq e(G_1)+e(G_2)+4+\frac{k_1^2+k_2^2+k_1+k_2}{2}+4 \\ &\leq e(G_1E_4^{(l)}G_2)+\frac{k_1^2+k_2^2+k_1+k_2}{2}+k_1k_2=e(G_1E_4^{(l)}G_2)+\frac{k^2+k}{2}. \end{aligned}$$

If $k_1k_2\leq 3$, without loss of generality, suppose that $k_1=1$, $k_2=3$ or $k_1=1$, $k_2=2$. If $k_1=1$ and $k_2=3$, then by Lemmas 2.1 and 2.4, we have

$$\begin{aligned} S_4(G_1E_4^{(l)}G_2) &\leq S_4(G_1\cup G_2)+S_4(E_4^{(l)})=\mu_1(G_1)+S_3(G_2)+8 \\ &\leq \mu_1(G_1)+e(G_2)+6+8=e(G_1)+e(G_2)+4+10+\mu_1(G_1)-e(G_1) \\ &\leq e(G_1E_4^{(l)}G_2)+10 \end{aligned}$$

for $l=1,2,\ldots,11$.

If $k_1=1$ and $k_2=2$, then by Lemmas 2.1, 2.2 and 2.4, we have

$$\begin{aligned} S_3(G_1E_4^{(l)}G_2) &\leq S_3(G_1\cup G_2)+S_3(E_4^{(l)}) \\ &\leq \mu_1(G_1)+e(G_2)+3+8=e(G_1)+e(G_2)+4+6+\mu_1(G_1)+1-e(G_1) \\ &\leq e(G_1E_4^{(l)}G_2)+6 \end{aligned}$$

for $l=1,2,\ldots,11$. □

Corollary 3.2. *If G is one of the following graphs, then Brouwer's conjecture holds for G:*

$$K_{n_1}E_1^{(1)}K_{n_2}E_1^{(1)}\cdots E_1^{(1)}K_{n_q},$$

$$K_{n_1}E_2^{(t)}K_{n_2}E_2^{(t)}\cdots E_2^{(t)}K_{n_q},$$

$$K_{n_1}E_3^{(j)}K_{n_2}E_3^{(j)}\cdots E_3^{(j)}K_{n_q},$$

$$K_{n_1}E_4^{(l)}K_{n_2}E_4^{(l)}\cdots E_4^{(l)}K_{n_q},$$

where $n_i \geq 4$ with $i = 1, 2, \ldots, q$, and $q \geq 2$, while $t = 1, 2$, $j = 1, 2, \ldots, 5$, and $l = 1, 2, \ldots, 11$.

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