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Research Article

On strongly connected Markov graphs of maps on combinatorial trees

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Abstract

Markov graphs form a special class of digraphs constructed from self-maps on the vertex sets of combinatorial trees. In this paper, the trees that admit cyclic permutations of their vertex sets with non-strongly connected Markov graphs in terms of the existence of a special subset of edges are characterized. Additionally, the structure of self-maps of finite sets, which produce strongly connected Markov graphs for all trees, is described. A similar question, concerning which self-maps produce strongly connected Markov graphs for some trees, is answered for the class of permutations.

Keywords: trees; permutations; Markov graphs; strongly connected digraphs.

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1. Introduction

Any vertex self-map σ on a combinatorial tree X gives rise to a special directed graph $\Gamma(X, \sigma)$, in which the vertices represent the edges of X and the arcs encode the covering relation between edges under σ. Specifically, the edge $uv \in E(X)$ covers the edge $xy \in E(X)$ under the map σ provided both x, y lie on a (unique) shortest path between $\sigma(u)$ and $\sigma(v)$ in X. The digraph $\Gamma(X, \sigma)$ is called the Markov graph for the pair (X, σ).

This construction stems from combinatorial dynamics, particularly from the problem of co-existence of periods of periodic points for continuous maps on topological trees (see the work of Bernhardt [1]). The central result in this area is the celebrated Sharkovsky theorem [9], which gives a complete answer to the mentioned problem in a rather unexpected and a beautiful way when the underlying space is the unit interval. It turns out that there is a linear ordering $\lhd$ of natural numbers (called Sharkovsky ordering) such that for any continuous map $f : [0, 1] \to [0, 1]$, the existence of a periodic point with period n implies the existence of periodic points of periods m for all $m \lhd n$, see [9]. Moreover, this result is optimal in a sense that for every m, there exists a continuous map f that has a periodic point of period m but no periodic points of periods n for all $m \lhd n$.

As it was shown by Ho and Morris in [2], Sharkovsky theorem can be proved using purely combinatorial arguments based on the careful examination of cycles in Markov graphs. Note that for the maps f of the unit interval to itself, the corresponding combinatorial tree X is just a path with the mentioned vertex self-map σ being its cyclic permutation. The graph-theoretic properties of these digraphs were studied by Pavlenko in [6–8].

In the present work, we study the structure of tree self-maps with strongly connected Markov graphs. The paper is organized as follows. After giving all the necessary definitions and preliminary results in Section 2, we present all the main results in Section 3. Namely, we briefly note that a cyclic permutation on a tree always produces a weakly connected Markov graph, which is not the case for strong connectedness. Then we give a criterion for trees X that admit cyclic permutations of their vertex sets with non-strongly connected Markov graphs (Theorem 3.1). In Proposition 3.1, we characterize the maps $\sigma\colon V \to V$ that produce strongly connected Markov graphs for all trees on V. The similar question of which maps produce strongly connected Markov graphs for some trees is answered for the class of permutations in Theorem 3.2.

2. Preliminaries

In this paper, we consider both undirected and directed graphs. All our graphs will be finite. Thus, a *graph* G is an ordered pair (V, E), where $V = V(G)$ is its *vertex set* and $E = E(G)$ is its *edge set* (the *edge* is simply an unordered pair of vertices). If there is an edge $e = uv \in E(G)$, then we say that the vertices u, v are *adjacent*, and both of them are *incident* with e.

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For a set of vertices $S \subset V(G)$, by $G[S]$ we denote the corresponding induced subgraph: $V(G[S]) = S$ and $E(G[S]) = \{uv \in E(G) : u, v \in S\}$. We also use the notation $E_G(S)$ for the edge set of $G[S]$. Similarly, for a set of edges $E' \subset E(G)$, by $G[E']$ we denote the corresponding induced subgraph: $V(G[E']) = \{u \in V(G) : u \text{ is incident with some edge } e \in E'\}$ and $E(G[E']) = E'$.

A set of vertices in a graph is *independent* provided no two of them are adjacent. The cardinality of the largest independent set of vertices in a graph G is called its *independence number*, denoted by $\alpha(G)$. The following upper bound for $\alpha(G)$ will be used in proving Theorem 3.2. A vertex in a (finite) graph is called a *cut vertex* if its deletion increases the number of connected components.

Theorem 2.1. [5, Theorem 2.2] *For a connected graph G with $n \geq 2$ vertices, the following inequality holds:*

$$\alpha(G) \leq n - \frac{c(G)+1}{2},$$

where $c(G)$ is the number of cut vertices in G.

Similar to independent sets of vertices, a set of edges is called a *matching* if no two edges in the set share a common vertex. A set of edges $E' \subset E(G)$ is *spanning* if every vertex in a graph is incident to some edge from this set, i.e., if $V(G[E']) = V(G)$. A spanning matching is called *perfect*.

The vertex set of any connected graph G is endowed with the natural metric d_G, where $d_G(u, v)$ equals the length of the shortest path between u and v. For a connected graph G and a pair of its vertices $u, v \in V(G)$, the set

$$[u, v]_G = \{x \in V(G) : d_G(u, x) + d_G(x, v) = d_G(u, v)\}$$

is called the *metric interval* between u and v. Similarly, for an edge $uv \in E(G)$, the sets

$$W_G(u, v) = \{x \in V(G) : d_G(x, u) < d_G(x, v)\} \quad \text{and} \quad W_G(v, u) = \{x \in V(G) : d_G(x, v) < d_G(x, u)\}$$

are called the *half-spaces* generated by uv.

A graph without cycles is called a *forest*. A *tree* is a connected forest. A vertex of degree one is called a *leaf*. For a tree X, by $\mathrm{Leaf}(X)$ we denote the set of all leaves in X. We also note that a tree can have at most one perfect matching.

For a set V, by $\mathrm{Tr}(V)$ we denote the class of all trees X with $V(X) = V$.

A *digraph* D is an ordered pair (V, A), where $V = V(D)$ is its *vertex set* and $A = A(D) \subset V \times V$ is its *arc set*. The arc $(u, v) \in A(D)$ will also be denoted simply as $u \to v$. A *loop* is an arc of the form $u \to u$.

We say that a vertex v is *reachable* from a vertex u if there is a directed walk from u to v, i.e., a finite sequence of vertices $x_1, \ldots, x_m \in V(D)$ with $u \to x_1 \to \cdots \to x_m \to v$.

The set $N_D^+(u) = \{v \in V(D) : (u, v) \in A(D)\}$ is called the *out-neighborhood* of a vertex $u \in V(D)$. A set of vertices $S \subset V(D)$ in a digraph D is called *closed* if $N_D^+(u) \subset S$ for all $u \in S$.

A digraph D is called *strongly connected* if every pair of vertices in D is reachable from each other. Note that strong connectedness of D is equivalent to each of the following two conditions: D contains a spanning closed walk, or D does not contain proper closed sets. A *strong component* in a digraph D is its maximal strongly connected subdigraph. The *condensation* of a digraph D is a digraph with vertices corresponding to the strong components of D, with an arc $D_1 \to D_2$ between two strong components D_1 and D_2 provided $u \to v$ in D for some $u \in V(D_1)$, $v \in V(D_2)$. It is easy to observe that the condensation is an acyclic digraph.

Let V be a finite set. We denote the full transformation semigroup on V by $\mathcal{T}(V)$, and the symmetric group on V by $\mathcal{S}(V)$. The identity map on V is denoted by id_V.

An element $x \in V$ is a *fixed point* for a map $\sigma \in \mathcal{T}(V)$ if $\sigma(x) = x$. By $\mathrm{fix}\,\sigma$ we denote the set of all fixed points for σ.

An element $x \in V$ is called a *periodic point* for a map $\sigma \in \mathcal{T}(V)$ if there is $k \in \mathbb{N}$ with $\sigma^k(x) = x$. The smallest such number k is called the *period of* x. For example, fixed points are just periodic points of period one. The set $\mathrm{orb}_\sigma(x) = \{x, \sigma(x), \ldots, \sigma^n(x), \ldots\}$ is called the *orbit of* x under σ. Hence, x is a σ-periodic point if and only if the restriction of σ to $\mathrm{orb}_\sigma(x)$ is a cyclic permutation.

Let X be a tree and $\sigma\colon V(X) \to V(X)$ be a map. The corresponding *Markov graph* $\Gamma = \Gamma(X, \sigma)$ is a directed graph with the vertex set $V(\Gamma) = E(X)$ and the arc set

$$A(\Gamma) = \{uv \to xy : x, y \in [\sigma(u), \sigma(v)]_X\}.$$

Thus, the vertices in Γ correspond to the edges of X, and the existence of an arc $uv \to xy$ means that the edge uv covers the edge xy under σ.

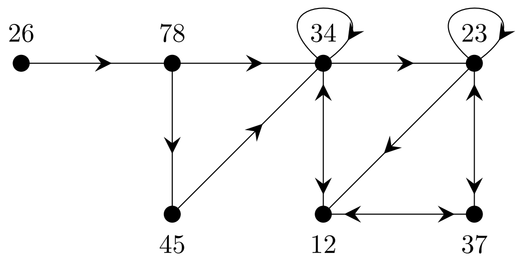

Figure 2.1: The Markov graph $\Gamma(X,\sigma)$ for the pair (X,σ) from Example 2.1.

Example 2.1. *Let X be a tree with the vertex set $V(X) = \{1,\dots,8\}$ and the edge set $E(X) = \{12,23,26,34,37,45,78\}$. Consider the map*

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 4 & 7 & 1 & 4 & 3 & 8 & 3 & 5 \end{pmatrix}$$

from $V(X)$ to itself. The corresponding Markov graph $\Gamma(X,\sigma)$ is depicted in Figure 2.1.

The following lemma is particularly useful for proving the existence of arcs in Markov graphs.

Lemma 2.1. [3, Lemma 3.17] *Let X be a tree and $\sigma\colon V(X) \to V(X)$ be a map. Then for every pair of vertices $u,v \in V(X)$ and an edge $e \in E_X([\sigma(u),\sigma(v)]_X)$, there exists an edge $e' \in E_X([u,v]_X)$ with $e' \to e$ in $\Gamma(X,\sigma)$.*

As mentioned earlier, our digraphs can have loops. In the case of Markov graphs $\Gamma(X,\sigma)$, the existence of a loop at $e = uv \in E(X)$ means that either $\sigma(u) \in W_G(u,v)$ and $\sigma(v) \in W_G(v,u)$, or $\sigma(v) \in W_G(u,v)$ and $\sigma(u) \in W_G(v,u)$. In the former case, the edge e will be called *σ-positive*, and in the latter case, the edge e is *σ-negative*. For illustration, in the pair (X,σ) from Example 2.1, the edge 34 is σ-positive and the edge 23 is σ-negative. Note also that all σ-negative edges form a matching in X.

By $p(X,\sigma)$ and $n(X,\sigma)$ we denote the numbers of σ-positive and σ-negative edges, respectively. These numbers are related to the number of fixed points by the following equality.

Theorem 2.2. [4, Theorem 4.2] *For any tree X and its map $\sigma\colon V(X) \to V(X)$, it holds that $n(X,\sigma) + |\operatorname{fix}\sigma| = p(X,\sigma) + 1$.*

3. Main results

It is easy to show that $\Gamma(X,\sigma)$ is weakly connected whenever σ is a cyclic permutation of $V(X)$. Indeed, Let us fix a tree X and its edge $uv \in E(X)$ with $u \in \operatorname{Leaf}(X)$. Then, for any edge $xy \in E(X) \setminus \{uv\}$, there exists $k \in \mathbb{N}$ with $\sigma^k(x) = u$. It is clear that there is an arc $xy \to uv$ in $\Gamma(X,\sigma^k)$. Hence, by Lemma 2.1 and a simple inductive argument, the edge uv is reachable from xy in $\Gamma(X,\sigma)$. This implies that $\Gamma(X,\sigma)$ is weakly connected. However, as the following example demonstrates, cyclic permutations of trees can have non-strongly connected Markov graphs.

Example 3.1. *Let X be a path with*

$$V(X) = \{1,\dots,6\} \text{ and } E(X) = \{12,23,34,45,56\}.$$

Then the cyclic permutation $\sigma = (135246)$ has a non-strongly connected Markov graph $\Gamma(X,\sigma)$: the sets of edges $\{12,34,56\}$ and $\{23,45\}$ induce two strong components in $\Gamma(X,\sigma)$ (see Figure 3.1).

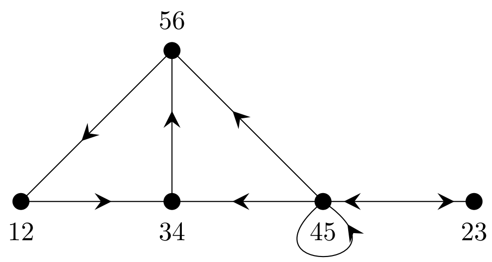

Figure 3.1: The Markov graph $\Gamma(X,\sigma)$ for the pair (X,σ) from Example 3.1.

It is not surprising that for a given tree, the existence of a cyclic permutation with a non-strongly connected Markov graph is directly related to the tree's structure. The next theorem fully characterizes such trees. In order to state this result, we need a specific definition: a forest is called *balanced* if each of its trees has the same number of vertices. Trivially, any single tree is a balanced forest.

Theorem 3.1. *For a tree X, there exists a cyclic permutation σ of $V(X)$ such that its Markov graph $\Gamma(X,\sigma)$ is not strongly connected if and only if there exists a proper (i.e., $E' \neq E(X)$) spanning set of edges $E' \subset E(X)$ such that $X[E']$ is a balanced forest.*

Proof. Necessity. Assume σ is a cyclic permutation of $V(X)$ with a non-strongly connected $\Gamma(X,\sigma)$. Let $E' \subset E(X)$ be the edge set which corresponds to a strong component of $\Gamma(X,\sigma)$ that is also closed in $\Gamma(X,\sigma)$ (i.e., E' corresponds to a vertex with zero out-degree in the condensation of $\Gamma(X,\sigma)$). Clearly, E' is a proper subset of $E(X)$.

Suppose E' is not spanning. Then there exists a vertex $u \in V(X)$ such that $E_X(u) \cap E' = \emptyset$. Fix an edge $vw \in E'$. Since σ is a cyclic permutation, $\sigma^k(v) = u$ for some $k \geq 1$. Trivially, $\sigma^k(w) \neq u$. Let $x \in [\sigma^k(v), \sigma^k(w)]_X = [u, \sigma^k(w)]_X$ be the vertex with $ux \in E(X)$. By Lemma 2.1, the edge ux is reachable from vw in $\Gamma(X,\sigma)$, contradicting the closedness of E'. Thus, E' must be spanning.

Now, let $X_1, \dots, X_m$ be the trees in the forest $X[E']$. Since E' is proper and spanning, we have $m \geq 2$. We aim to prove that for every $1 \leq i \leq m$, there exists $1 \leq j \leq m$ such that $\sigma(V(X_i)) \subset V(X_j)$.

Indeed, because E' is spanning, we have $V(X) = \bigsqcup_{i=1}^{m} V(X_i)$. This implies that for all pairs of vertices $x, y \in V(X_i)$, there exist indices $1 \leq j, k \leq m$ such that $\sigma(x) \in V(X_j)$ and $\sigma(y) \in V(X_k)$.

We aim to prove that $j = k$. Assume, for contradiction, that $j \neq k$. Then there is an edge $e \in E_X([\sigma(x), \sigma(y)]_X)$ such that $e \notin E'$. Since $V(X_i)$ is connected, by Lemma 2.1, there is $e' \in E_X([x,y]_X) \subset E(X_i)$ with $e' \to e$. This contradicts the assumption that E' is a strong component in $\Gamma(X,\sigma)$. Therefore, $j = k$, as required.

Thus, fix $1 \leq i, j \leq m$ with $\sigma(V(X_i)) \subset V(X_j)$. Clearly, $|V(X_i)| \leq |V(X_j)|$ as σ is bijective. Assume, for the sake of contradiction, that $|V(X_i)| < |V(X_j)|$. Since $V(X_j)$ is connected, there must exist a vertex $z \in V(X_j) \setminus \sigma(V(X_i))$ that is adjacent to some vertex $t \in \sigma(V(X_i))$.

We have $\sigma^{-1}(z) \in V(X_k)$ for some $k \neq i$. Since σ is a cyclic permutation, it follows that $\sigma^{n-1}(z) = \sigma^{-1}(z)$ and $\sigma^{n-1}(t) = \sigma^{-1}(t)$. However, $E_X([\sigma^{n-1}(z), \sigma^{n-1}(t)]_X)$ must include edges from $E(X) \setminus E'$ since $k \neq i$. Invoking Lemma 2.1, we again encounter a similar contradiction. Thus, $|V(X_i)| = |V(X_j)|$. This implies

$$|V(X_1)| = \cdots = |V(X_m)|$$

as the permutation σ is cyclic (otherwise, there would be a proper σ-invariant set in X). Therefore, the necessity of the condition is proved.

Sufficiency. Assume $E' \subset E(X)$ is a proper spanning set of edges such that the forest $X[E']$ is balanced. Let $X_1, \dots, X_m$ are the trees in $X[E']$. Trivially, $m \geq 2$. Since $X[E']$ is balanced, $|V(X_1)| = \cdots = |V(X_m)| = k$ for some $k \geq 1$. Define the vertex set $V(X_i) = \{x_i^1, \dots, x_i^k\}$ for each $1 \leq i \leq m$. Put

$$\sigma(x_i^j) = \begin{cases} x_{i+1}^j \text{ if } 1 \leq i \leq m-1, \\ x_1^{j+1} \text{ if } i = m \text{ and } 1 \leq j \leq k-1, \\ x_1^1 \text{ if } i = m \text{ and } j = k \end{cases}$$

for all $1 \leq i \leq m$ and $1 \leq j \leq k$. Since E' is spanning, σ is a correctly defined map on $V(X)$. Moreover, it is easy to see that σ is a cyclic permutation of $V(X)$.

Now, let $e' \in E'$. Since $e' \in E(X_i)$ for some $1 \leq i \leq m$, we have $e' = x_i^{j_1} x_i^{j_2}$, where $1 \leq j_1, j_2 \leq k$. If $i \neq m$, then

$$[\sigma(x_i^{j_1}), \sigma(x_i^{j_2})]_X = [x_{i+1}^{j_1}, x_{i+1}^{j_2}]_X \subset V(X_{i+1}).$$

If $i = m$ and $j_1, j_2 \neq k$, then

$$[\sigma(x_i^{j_1}), \sigma(x_i^{j_2})]_X = [x_1^{j_1+1}, x_1^{j_2+1}]_X \subset V(X_1).$$

Finally,

$$[\sigma(x_m^{j_1}), \sigma(x_m^k)] = [\sigma(x_i^{j_1}), \sigma(x_i^{j_2})]_X = [x_1^{j_1+1}, x_1^1]_X \subset V(X_1)$$

for $i = m$, $j_1 \neq k$ and $j_2 = k$ (similarly, one consider the case $i = m$, $j_1 = k$ and $j_2 \neq k$). This means that E' is a proper closed set in $\Gamma(X,\sigma)$, implying that $\Gamma(X,\sigma)$ is not strongly connected. □

We illustrate the construction of a cyclic permutation from the proof of sufficiency in Theorem 3.1 with the following example.

Example 3.2. *Consider a tree X with $V(X) = \{0, \ldots, 9, a, b\}$ and $E(X) = \{01, 02, 03, 04, 45, 46, 67, 68, 89, 9a, 9b\}$.*

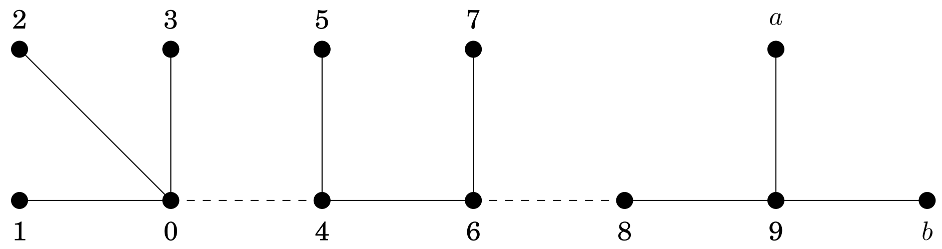

Figure 3.2: The tree X in Example 3.2, with two dashed edges, 04 and 68, that do not belong to the spanning edge set E'.

Put $E' = E(X) \setminus \{04, 68\}$ (see Figure 3.2). Clearly, E' is spanning. Also, the induced forest $X[E']$ consists of three trees, each having four vertices, which makes it balanced. The corresponding trees X_1, X_2, X_3 have vertex sets $V(X_1) = \{0, 1, 2, 3\}$, $V(X_2) = \{4, 5, 6, 7\}$, and $V(X_3) = \{8, 9, a, b\}$. In the notation of Theorem 3.1, we have $m = 3$, $k = 4$. Further, put $x_1^1 = 0$, $x_1^2 = 1$, $x_1^3 = 2$, and so on, ending with $x_3^3 = a$, $x_3^4 = b$.

Then the cyclic permutation

$$\sigma = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & a & b \\ 4 & 5 & 6 & 7 & 8 & 9 & a & b & 1 & 2 & 3 & 0 \end{pmatrix} = (04815926a37b)$$

has a non-strongly connected Markov graph $\Gamma(X, \sigma)$. Indeed, E' is a proper closed set in $\Gamma(X, \sigma)$ (alternatively, one can observe that the edge set $E(X) \setminus E' = \{04, 68\}$ induces a strong component). Hence, $\Gamma(X, \sigma)$ is not strongly connected.

Corollary 3.1. *Assume that for every cyclic permutation σ of the vertex set of a tree X, its Markov graph $\Gamma(X, \sigma)$ is strongly connected. Then X does not have a perfect matching, and $|W_X(u, v)| \neq |W_X(v, u)|$ for all edges $uv \in E(X)$.*

Proof. If E' is a perfect matching in a tree X, then E' is spanning, and $X[E']$ is a balanced forest (each tree in $X[E']$ is an edge and thus has two vertices). Similarly, if $|W_X(u, v)| = |W_X(v, u)| = k$ for some edge $uv \in E(X)$, then put $E' = E(X) \setminus \{uv\}$. Clearly, E' is spanning, and the forest $X[E']$ is balanced as it consists of two trees with k vertices. Hence, in each case, Theorem 3.1 guarantees the existence of a cyclic permutation of $V(X)$ with a non-strongly connected Markov graph. □

Corollary 3.2. *If the number of vertices in a tree X is prime, then $\Gamma(X, \sigma)$ is strongly connected for every cyclic permutation σ of $V(X)$.*

Proof. This follows directly from Theorem 3.1. □

Using Corollary 3.2, we can fully characterize the dynamical structure of maps that have strongly connected Markov graphs for all trees.

Proposition 3.1. *Let $n = |V| \geq 3$. For a map $\sigma \in \mathcal{T}(V)$, the Markov graph $\Gamma(X, \sigma)$ is strongly connected for all trees $X \in \mathrm{Tr}(V)$ if and only if n is a prime number and σ is a cyclic permutation.*

Proof. Sufficiency. Follows immediately from Corollary 3.2.

Necessity. Assume that the Markov graph $\Gamma(X, \sigma)$ is strongly connected for all trees $X \in \mathrm{Tr}(V)$. First, suppose σ is not a permutation. In this case, there exist distinct elements $u, v \in V$ with $\sigma(u) = \sigma(v)$. Then for any tree $X \in \mathrm{Tr}(V)$ with $uv \in E(X)$, the edge uv will have zero out-degree in $\Gamma(X, \sigma)$. Since $n \geq 3$, we have $|E(X)| \geq 2$. Thus, in this case, $\Gamma(X, \sigma)$ is not strongly connected.

Hence, let σ be a permutation of V. From the inequality $|E(X)| \geq 2$, it follows that $\sigma \neq \mathrm{id}_V$. If σ is not a cyclic permutation, then there exists a proper σ-invariant subset $V' \subset V$ with $|V'| \geq 2$. Choose a pair of trees $X_1 \in \mathrm{Tr}(V')$ and $X_2 \in \mathrm{Tr}(V \setminus V')$, and select vertices $u \in V(X_1)$, $v \in V(X_2)$. Consider the new tree $X \in \mathrm{Tr}(V)$ with $E(X) = E(X_1) \cup E(X_2) \cup \{uv\}$. It is clear that $\Gamma(X, \sigma)$ is not strongly connected, as $E(X_1)$ is a proper closed set in $\Gamma(X, \sigma)$. Therefore, σ is a cyclic permutation of V.

Finally, assume n is not a prime number, so $n = mk$ for some $m, k \geq 2$. Fix an element $u \in V$. Since σ is a cyclic permutation, we have $\mathrm{orb}_\sigma(u) = V$. For all $0 \leq i \leq m-1$ and $0 \leq j \leq k-1$, define the number $g(j, i) = jm + i$. Additionally, define $v_i = \sigma^i(u)$ for each $0 \leq i \leq n-1$.

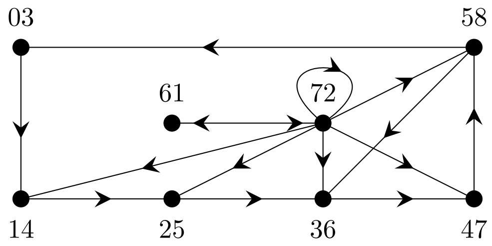

Figure 3.3: The Markov graph $\Gamma(X, \sigma)$ for the pair (X, σ) from Example 3.3.

Consider a graph X on V with the edge set

$$E(X) = \bigcup_{i=0}^{m-1} \left(\{v_{g(j,i)}v_{g(j+1,i)} : 0 \le j \le k-2\} \cup \{v_{g(k-1,i)}v_{i+1}\}\right) \setminus \{v_{mk-1}v_m\}.$$

It can be observed that $X \in \mathrm{Tr}(V)$ is a path on V. Also, the edge set $\bigcup_{i=0}^{m-1}\{v_{g(j,i)}v_{g(j+1,i)} : 0 \le j \le k-2\}$ is closed in $\Gamma(X,\sigma)$ (in fact, it induces a strong component). Indeed, for an edge $v_{g(j,i)}v_{g(j+1,i)}$ with $i \ne m-1$ and $j \ne k-2$, we have

$$\sigma(v_{g(j,i)}) = \sigma(\sigma^{g(j,i)}(u)) = \sigma^{jm+i+1}(u) = v_{g(j,i+1)};$$
$$\sigma(v_{g(j+1,i)}) = \sigma(\sigma^{g(j+1,i)}(u)) = \sigma^{(j+1)m+i+1}(u) = v_{g(j+1,i+1)}.$$

Thus, $N^+_{\Gamma(X,\sigma)}(v_{g(j,i)}v_{g(j+1,i)}) = \{v_{g(j,i+1)}v_{g(j+1,i+1)}\}$. In case of $i = m-1$ and $j = k-2$, we observe

$$\sigma(v_{g(k-2,m-1)}) = \sigma(\sigma^{(k-2)m+m-1}(u)) = \sigma^{(k-1)m}(u) = v_{g(k-1,0)};$$
$$\sigma(v_{g(k-1,m-1)}) = \sigma(\sigma^{(k-1)m+m-1}(u)) = \sigma^{mk}(u) = u = v_{g(0,0)}.$$

Hence, $N^+_{\Gamma(X,\sigma)}(v_{g(k-2,m-1)}v_{g(k-1,m-1)}) = \{v_{g(j,0)}v_{g(j+1,0)} : 0 \le j \le k-2\}$. The obtained contradiction proves that n is a prime number. □

The next example illustrates the construction of a tree X for a given cyclic permutation from the proof of necessity part in Proposition 3.1.

Example 3.3. *Let $n = 9$, $V = \{0,\dots,8\}$, and $\sigma = (012345678)$ be a cyclic permutation of V. We have $m = k = 3$. Put $u = 0$ and use the construction from Proposition 3.1 to define a path $X \in \mathrm{Tr}(V)$ with the edge set $E(X) = \{03, 36, 16, 14, 47, 27, 25, 58\}$. Then the set of edges $\{03, 36, 14, 47, 25, 58\}$ induces a strong component in $\Gamma(X,\sigma)$. This indicates that $\Gamma(X,\sigma)$ is not strongly connected (see Figure 3.3).*

Now we describe the structure of permutations that produce strongly connected Markov graphs for some trees.

Theorem 3.2. *Let $n = |V| \ge 5$. For a permutation $\sigma \in \mathcal{S}(V)$, there exists a tree $X \in \mathrm{Tr}(V)$ such that $\Gamma(X,\sigma)$ is strongly connected if and only if $|\operatorname{fix}\sigma| \le \frac{n-1}{2}$.*

Proof. Necessity. If $\Gamma(X,\sigma)$ is strongly connected for some tree $X \in \mathrm{Tr}(V)$, then, since σ is a permutation, we must have $\operatorname{fix}\sigma \subset V(X) \setminus \mathrm{Leaf}(X)$ (otherwise, if there exists a fixed point which is a leaf vertex, then the corresponding unique edge would induce a strong component in $\Gamma(X,\sigma)$).

Similarly, having two adjacent fixed points for σ will produce an edge which would be a singleton strong component in $\Gamma(X,\sigma)$. Further, the inequality $n \ge 5$ implies that $\operatorname{fix}\sigma$ is an independent set of vertices in X.

Define $l = |\mathrm{Leaf}(X)|$ and $l' = |\mathrm{Leaf}(X \setminus \mathrm{Leaf}(X))|$. Then clearly $l' \le l$ and $c(X \setminus \mathrm{Leaf}(X)) = n - l - l'$ (recall that $c(G)$ denotes the number of cut vertices in a graph G). By Theorem 2.1, we have

$$\alpha(X \setminus \mathrm{Leaf}(X)) \le |V(X) \setminus \mathrm{Leaf}(X)| - \frac{c(X \setminus \mathrm{Leaf}(X)) + 1}{2} = n - l - \frac{n-l-l'+1}{2} = \frac{n-l+l'-1}{2} \le \frac{n-1}{2},$$

which implies $|\operatorname{fix}\sigma| \le \frac{n-1}{2}$.

Sufficiency. Here we consider several cases.

Case 1. $\operatorname{fix}\sigma = \emptyset$ and $\sigma^2 = \mathrm{id}_V$.
In this case, n is even, so $n \ge 6$. Let $u_1, \dots, u_{\frac{n}{2}} \in V$ be the elements with pairwise disjoint orbits. Define $v_i = \sigma(u_i)$ for $1 \le i \le \frac{n}{2}$. Consider a graph X on V with the edge set $E(X) = \{u_iv_{i-2}, u_iv_{i-1} : 3 \le i \le \frac{n}{2}\} \cup \{u_1u_2, u_2v_1, v_{\frac{n}{2}-1}v_{\frac{n}{2}}\}$.

It is easy to see that $X \in \mathrm{Tr}(V)$ is a path on V. Moreover, the Markov graph $\Gamma(X,\sigma)$ is strongly connected as it contains a spanning strongly connected subdigraph Γ' with (see Figure 3.4):

$$\begin{aligned} A(\Gamma') =&\{(u_iv_{i-2}, u_iv_{i-1}), (u_iv_{i-2}, u_{i-1}v_{i-2}), (u_iv_{i-1}, u_iv_{i-2}), (u_iv_{i-1}, u_{i+1}v_{i-1}) : 3 \le i \le \frac{n}{2} - 1\} \\ &\cup\{(u_1u_2, u_3v_1), (u_2v_1, u_1u_2), (u_2v_1, u_3v_1), (v_{\frac{n}{2}-1}v_{\frac{n}{2}}, u_{\frac{n}{2}}v_{\frac{n}{2}-2}), (u_{\frac{n}{2}}v_{\frac{n}{2}-1}, v_{\frac{n}{2}-1}v_{\frac{n}{2}}), \\ &(u_{\frac{n}{2}}v_{\frac{n}{2}-1}, u_{\frac{n}{2}}v_{\frac{n}{2}-2}), (u_{\frac{n}{2}}v_{\frac{n}{2}-2}, u_{\frac{n}{2}}v_{\frac{n}{2}-1}), (u_{\frac{n}{2}}v_{\frac{n}{2}-2}, u_{\frac{n}{2}-1}v_{\frac{n}{2}-2})\}. \end{aligned}$$

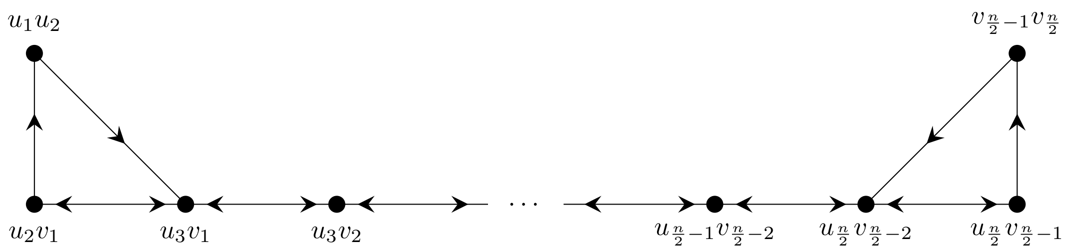

Figure 3.4: The spanning strongly connected subdigraph Γ'.

Case 2. $\mathrm{fix}\,\sigma = \emptyset$.
Here we use induction on $n \ge 5$. For the base case, we assume that $n = 5$ and $V = \{1, 2, 3, 4, 5\}$. Considering the possible cycle types, the permutation σ can be one of the following: $\sigma_1 = (12)(345)$ or $\sigma_2 = (12345)$. Consider a path X_1 on V with $E(X_1) = \{13, 14, 24, 25\}$. One can check by hand that each Markov graph $\Gamma(X_1, \sigma_i)$, $1 \le i \le 2$, is strongly connected. Thus, the base case holds.

Now we proceed with the induction step. Let $n \ge 6$. If the period of each σ-periodic point is bounded by 2, then $\sigma^2 = \mathrm{id}_V$, a case we have already considered.

Hence, assume there exists a σ-periodic point whose period is at least 3. If σ is a cyclic permutation, then for a path $X \in \mathrm{Tr}(V)$ with $E(X) = \{\sigma^i(u)\sigma^{i+1}(u) : 0 \le i \le n-1\}$, the Markov graph $\Gamma(X,\sigma)$ is strongly connected. Thus, suppose that σ is not a cyclic permutation.

Subcase 2.1. For any σ-periodic point $u \in V$ it holds $|V \setminus \mathrm{orb}_\sigma(u)| \le 4$.
In this case, the possible cycle types for σ are: $\sigma_3 = (12)(3456)$, $\sigma_4 = (123)(456)$, $\sigma_5 = (123)(4567)$, $\sigma_6 = (1234)(5678)$.

Let $V(X_2) = V(X_1) \cup \{6\}$, $E(X_2) = E(X_1) \cup \{56\}$; $V(X_3) = V(X_2) \cup \{7\}$, $E(X_3) = E(X_2) \cup \{67\}$; $V(X_4) = V(X_3) \cup \{8\}$, $E(X_4) = \{18, 12, 23, 35, 56, 67, 74\}$. It is easy to see that all graphs X_i for $2 \le i \le 4$ are paths. Moreover, the Markov graphs $\Gamma(X_2, \sigma_3)$, $\Gamma(X_2, \sigma_4)$, $\Gamma(X_3, \sigma_5)$, and $\Gamma(X_4, \sigma_6)$ are strongly connected.

Subcase 2.2. There exists a σ-periodic point $u \in V$ (with period $m \ge 3$) such that $|V \setminus \mathrm{orb}_\sigma(u)| \ge 5$.
Consider the set $V' = V \setminus \mathrm{orb}_\sigma(u)$ and the restriction $\sigma' = \sigma|_{V'}$. By the induction assumption, there exists a tree $X' \in \mathrm{Tr}(V')$ such that $\Gamma(X', \sigma')$ is strongly connected.

Since σ is not a cyclic permutation and $\mathrm{fix}\,\sigma = \emptyset$, it follows that $|V'| \ge 2$, implying $E(X') \ne \emptyset$. Fix an edge $xy \in E(X')$, and consider a new graph X on V with the edge set

$$E(X) = (E(X') \setminus \{xy\}) \cup \{ux, uy, \sigma(u)x\} \cup \{\sigma(u)\sigma^i(u) : 2 \le i \le m-1\},$$

which is partially depicted in Figure 3.5. Clearly, $X \in \mathrm{Tr}(V)$ is a tree on V. We want to prove that $\Gamma(X,\sigma)$ is also strongly connected. To do this, fix a spanning closed walk $W = \{e_1 \to \cdots \to e_{n-m-1} \to e_1\}$ in $\Gamma(X', \sigma')$. We can assume that $e_1 = xy$. Since $[\sigma(x), \sigma(y)]_X \subset [\sigma(x), \sigma(u)]_X \cup [\sigma(u), \sigma(y)]_X$, we have $e_2 \in E_X([\sigma(x), \sigma(u)]_X)$ or $e_2 \in E_X([\sigma(u), \sigma(y)]_X)$. If $e_2 \in E_X([\sigma(x), \sigma(u)]_X)$, then

$$ux \to e_2 \to \cdots \to e_{n-m-1} \to uy \to \sigma(u)x \to \sigma(u)\sigma^2(u) \to \ldots \cdots \to \sigma(u)\sigma^{m-1}(u) \to ux$$

is a spanning closed walk in $\Gamma(X,\sigma)$. Thus, let $e_2 \in E_X([\sigma(u), \sigma(y)]_X)$. Since $\mathrm{fix}\,\sigma = \emptyset$, we have $[x, \sigma(x)]_{X'} \ne \emptyset$. Fix an edge $e_k \in E_X([x, \sigma(x)]_{X'}) \subset E_X([\sigma(x), \sigma(u)]_X)$. In this case,

$$\begin{gathered} uy \to e_2 \to \cdots \to e_{n-m-1} \to ux \to \sigma(u)x \to \sigma(u)\sigma^2(u) \to \ldots \\ \cdots \to \sigma(u)\sigma^{m-1}(u) \to ux \to e_k \to \cdots \to e_{n-m-1} \to uy \end{gathered}$$

is a spanning closed walk in $\Gamma(X,\sigma)$. Hence, $\Gamma(X,\sigma)$ is strongly connected.

Now we are ready to tackle the general case of the theorem.

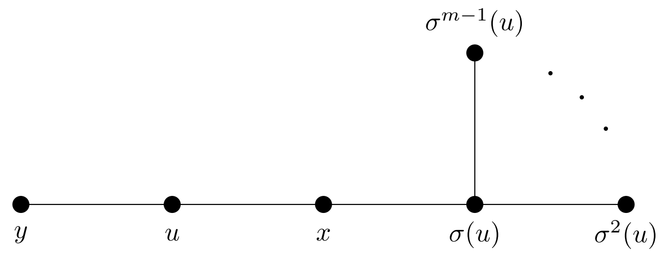

Figure 3.5: A fragment of the tree X used in the proof of Theorem 3.2.

Case 3. $|\operatorname{fix}\sigma| \leq \frac{n-1}{2}$.
Consider the set $V' = V \setminus \operatorname{fix}\sigma$ and the map $\sigma' = \sigma|_{V'}$. Since $\operatorname{fix}\sigma' = \emptyset$, there exists a tree $X' \in \operatorname{Tr}(V')$ such that $\Gamma(X', \sigma')$ is strongly connected. We have $|E(X')| = |V'| - 1 = n - |\operatorname{fix}\sigma| - 1 \geq \frac{n-1}{2}$. This implies that there exists an injective map $\varphi\colon \operatorname{fix}\sigma \to E(X')$. Consider a graph X on V with the edge set $E(X) = (E(X') \setminus \operatorname{Im}\varphi) \cup \{xu_x, xv_x : x \in \operatorname{fix}\sigma\}$, where $\varphi(x) = u_x v_x$ for all $x \in \operatorname{fix}\sigma$. One can think of X as being obtained from X' by subdividing each edge $\varphi(x)$ with the new vertex x.

We want to prove that $\Gamma(X, \sigma)$ is also a strongly connected digraph. It is sufficient to show this for the case where $|\operatorname{fix}\sigma| = 1$. Thus, let $\operatorname{fix}\sigma = \{x\}$. Fix a spanning closed walk $e_1 \to \cdots \to e_{n-2} \to e_1$ in $\Gamma(X', \sigma')$ with $e_1 = \varphi(x)$. Without loss of generality, assume that $e_2 \in E_X([\sigma(x), \sigma(u_x)]_X)$ (the case where $e_2 \in E_X([\sigma(x), \sigma(v_x)]_X)$ is considered similarly). Since $\operatorname{fix}\sigma' = \emptyset$, it follows that $v_x \neq \sigma(v_x)$, and thus, we can fix an edge $e_k \in E_X([\sigma(x), \sigma(v_x)]_X)$. In this case,

$$xu_x \to e_2 \to \cdots \to e_m \to xv_x \to e_k \to \cdots \to e_m \to xu_x$$

is a spanning closed walk in $\Gamma(X, \sigma)$. Thus, $\Gamma(X, \sigma)$ is strongly connected. In the case $|\operatorname{fix}\sigma| \geq 2$, we proceed inductively, subdividing each time an edge not incident to a fixed point by a new fixed point. □

The following examples provide two instances where the sufficiency of the condition in Theorem 3.2 fails: when $n = 4$ or when σ is not a permutation.

Example 3.4. *Let $V = \{1, 2, 3, 4\}$ and $\sigma = (12)(34)$. Clearly, the permutation σ does not have fixed points. However, for every tree $X \in \operatorname{Tr}(V)$, the Markov graph $\Gamma(X, \sigma)$ is not strongly connected. Indeed, there are exactly two non-isomorphic trees with four vertices: the path and the star. Note that if either 12 or 34 is an edge in X, then this edge will induce a strong component in $\Gamma(X, \sigma)$, making stars irrelevant for our consideration. If X is a path on V, then, without loss of generality, we can assume that $E(X) = \{13, 23, 24\}$ (as 12 and 34 cannot be edges in X). But in this case, the edge 23 induces a strong component in $\Gamma(X, \sigma)$.*

Example 3.5. *Let $V = \{1, 2, 3, 4, 5\}$ and $\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 2 & 1 & 1 & 2 \end{pmatrix}$. Then σ is not a permutation of V, but $|\operatorname{fix}\sigma| = 2 = \frac{n-1}{2}$. Our goal is to show that $\Gamma(X, \sigma)$ is not strongly connected for all trees $X \in \operatorname{Tr}(V)$. At first, note that $V \setminus \operatorname{Im}\sigma = \{3, 4, 5\}$. If at least one of these vertices is a leaf in $X \in \operatorname{Tr}(V)$, then the unique corresponding edge will induce a strong component in $\Gamma(X, \sigma)$. Thus, we consider the case where $\operatorname{Leaf}(X) = \{1, 2\}$, meaning that X is a 5-vertex path. In this case, at least one pair of the vertices $1, 2, 3$ will be adjacent. Then the corresponding edge will have zero out-degree in $\Gamma(X, \sigma)$, which means that the Markov graph is not strongly connected.*

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