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Research Article

Products of general Fibonomial coefficients via matrices

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Abstract

As a sequential analogue of binomial coefficients, we consider general Fibonomial coefficients with indices in arithmetic progressions. We give a recurrence relation and a generating matrix for the products of these coefficients. We find explicitly the spectrum of the generating matrix by constructing new relationships between the coefficients and characteristic polynomials of general Pascal matrices. We derive various identities for the general Fibonomial coefficients. Finally, we present a matrix approach to derive a formula for sums of these coefficients.

Keywords: Fibonomial coefficients; generating matrix; recurrence relation; generating function; sums.

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1. Introduction

For $n > 0$ and a nonzero integer A, the general Fibonacci and Lucas sequences are defined by $U_{n+1} = AU_n + U_{n-1}$ and $V_{n+1} = AV_n + V_{n-1}$, where $U_0 = 0,\ U_1 = 1$ and $V_0 = 2,\ V_1 = A$, respectively. When $A = 1$, the general Fibonacci and Lucas sequences $\{U_n, V_n\}$ are reduced to the usual Fibonacci and Lucas sequences $\{F_n, L_n\}$. The general Fibonomial coefficients (GFCs) are formed by terms of the sequence $\{U_n\}$ as follows for $n \geq m \geq 1$ and $r \geq 1$

$$\begin{Bmatrix} n \\ m \end{Bmatrix}_{U;r} = \frac{U_r U_{2r} \dots U_{rn}}{U_r U_{2r} \dots U_{r(n-m)} \cdot U_r U_{2r} \dots U_{rm}},$$

with $\left\{\begin{smallmatrix} n \\ n \end{smallmatrix}\right\}_{U;r} = \left\{\begin{smallmatrix} n \\ 0 \end{smallmatrix}\right\}_{U;r} = 1$ and 0 otherwise. When $r = 1$, the general Fibonomial coefficients $\left\{\begin{smallmatrix} n \\ m \end{smallmatrix}\right\}_{U,1}$ are reduced to the general Fibonacci coefficients denoted by $\left\{\begin{smallmatrix} n \\ m \end{smallmatrix}\right\}_{U}$. When also $A = 1$, the general Fibonomial coefficients $\left\{\begin{smallmatrix} n \\ m \end{smallmatrix}\right\}_{U}$ are reduced to the Fibonacci coefficients $\left\{\begin{smallmatrix} n \\ m \end{smallmatrix}\right\}_{F}$. The Fibonomial coefficients, which are sequential variants of binomial coefficients, have attracted a lot of attention with their interesting properties and connections with other known mathematical structures; for more details, see [4, 6, 8, 11, 13–16, 18, 19].

For $n > 0$ and a nonzero integer A, the $n \times n$ right-adjusted general Pascal matrix $P_n(A)$ is a matrix whose (i, j) entry is of the form

$$(P_n(A))_{ij} = \binom{j-1}{j+i-n-1} A^{i+j-n-1},$$

which is reduced to the right-adjusted Pascal matrix denoted by P_n for $A = 1$.

There are interesting relationships between the general Pascal matrix and the general Fibonomial coefficients; for details, see [1, 2, 12, 17].

Denote the roots of the characteristic equation $x^2 - Ax - 1 = 0$ of $\{U_n\}$ by α and β. From [12], we have that for $n, r \geq 1$,

$$\begin{aligned} U_{rn} &= V_r U_{r(n-1)} + (-1)^{r+1} U_{r(n-2)}, \\ V_{rn} &= V_r V_{r(n-1)} + (-1)^{r+1} V_{r(n-2)}. \end{aligned} \tag{1}$$

The $n \times n$ right-adjusted Pascal matrix P_n is defined as

$$P_n = \left[\binom{i-1}{n-j} \right]_{1 \leq i,j \leq n}.$$

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By considering the general Lucas sequence $\{V_m\}$, the authors of [12] defined the $n\times n$ general right-adjusted Pascal matrix $P_n(V_r)$ as

$$P_n(V_r)=\left[(-1)^{(r+1)(n-j)}V_r^{i+j-n-1}\binom{i-1}{n-j}\right]_{1\le i,j\le n}$$

and then show that for all $m>0$,

$$\boldsymbol{tr}\,(P_n^m(V_r))=\frac{U_{rnm}}{U_r}.$$

Observe that $P_n(V_1)=P_n(A)$. For $r=1$, $V_1=A$ and when $A=1$, the general Lucas sequence $\{V_n\}$ is reduced to the Lucas sequence $\{L_n\}$, and so the general Pascal matrix $P_n(V_r)$ is reduced to the Pascal matrix P_n. The authors of [12] explicitly derived that all the eigenvalues of $P_{n+1}(V_r)$ are

$$\alpha^{rn},\alpha^{r(n-1)}\beta^r,\ldots,\alpha^r\beta^{r(n-1)},\beta^{rn} \tag{2}$$

and the characteristic polynomial of $P_{n+1}(V_r)$ is

$$\mathcal{P}_n(V_r;x)=\prod_{j=0}^{n-1}\left(x-\alpha^{jr}\beta^{(n-j-1)r}\right)=\sum_{t=0}^{n}(-1)^{r\binom{t}{2}+t}\left\{{n \atop t}\right\}_{U,r}x^{n-t}, \tag{3}$$

where $\left\{{n \atop k}\right\}_{U;r}$ is the general Fibonomial coefficient. For $r=1$, $V_1=A$ and so we choose $A=1$, then $\mathcal{P}_n(V_1;x)$ is reduced to the characteristic polynomial of the Pascal matrix denoted by $\mathcal{P}_n(x)$.

Matrix methods are important and very convenient tools to solve problems stemming from number theory (see [9,10]). Our purpose in this paper is to derive generating matrices for the products of two general Fibonomial coefficients $\left\{{n \atop m}\right\}_{U,r}$ and derive recurrence relations for them. Further, by using matrix methods, we obtain new identities for them and explicit formulas for their sums.

2. GFCs with indices in arithmetic progress

For $1\le m\le k+1$, we derive a recursion and generating matrix for products of two general Fibonomial coefficients of the form

$$a_{n,m}:=s(m)\left\{{n+k \atop k-m+1}\right\}_{U,r}\times\left\{{n+m-2 \atop m-1}\right\}_{U,r},$$

where the sign function is defined as

$$s(m)=\begin{cases}(-1)^{\binom{m-1}{2}} & \text{if } r \text{ is odd},\\ (-1)^{m+1} & \text{if } r \text{ is even},\end{cases}$$

and $\left\{{n \atop m}\right\}_{U,r}$ stands for the general Fibonomial coefficients. Consider single general Fibonomial coefficients of the form

$$a_{1,m}=s(m)\left\{{k+1 \atop m}\right\}_{U,r}$$

by choosing $n=1$ in the definition of $a_{n,m}$.

We recall the identity $F_{n+m}=F_{m-1}F_n+F_mF_{n+1}$ (see pp. 176 of [20]). For the sequence $\{U_n\}$ and $k,m,n\in\mathbb{Z}$, and positive integer r, an analogue of this identity with indices in an arithmetic progression is

$$U_{rk}U_{r(n+m)}=U_{rm}U_{r(n+k)}+(-1)^{rm}U_{rn}U_{r(k-m)}. \tag{4}$$

Now, we give the following result.

Lemma 2.1. *For $n>0$, $1\le i\le k$ and for odd r,*

$$a_{1,i}a_{n,1}+(-1)^{i-1}a_{n,i+1}=a_{n+1,i},$$

and for even r,

$$a_{1,i}a_{n,1}-a_{n,i+1}=a_{n+1,i},$$

where $a_{n,i}$ is defined as before.

Proof. We consider the case when r is even. If we simplify the identity $a_{1,i}a_{n,1}-a_{n,i+1}=a_{n+1,i}$, we need to prove that

$$U_{r(k+1)}U_{r(n+i)}+U_{rn}U_{r(k-i+1)}=U_{ri}U_{r(n+k+1)}.$$

By taking $k\to k+1$ in (4), the last equality can be obtained. For the case when r is odd, the proof can be found in [11]. □

For $k, r \geq 1$, define the companion matrix G and the matrix H_n of order-$(k+1)$ as follows:

$$G = \begin{bmatrix} a_{1,1} & a_{1,2} & a_{1,3} & \cdots & a_{1,k+1} \\ 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & 1 & 0 \end{bmatrix} \text{ and } H_n = \begin{bmatrix} a_{n,1} & a_{n,2} & \cdots & a_{n,k+1} \\ a_{n-1,1} & a_{n-1,2} & \cdots & a_{n-1,k+1} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n-k,1} & a_{n-k,2} & \cdots & a_{n-k,k+1} \end{bmatrix}. \quad (5)$$

The matrix G is referred to as the *general Fibonomial matrix* since its powers generate the products of two general Fibonomial coefficients. Now, we give our first main result.

Theorem 2.1. *For $n > 0$, $H_n = G^n$.*

Proof. By the definition of H_n and the general Fibonomial coefficients, the proof is followed for $n = 1$. Suppose that the equality holds for $n \geq 1$. Now, we show that the equation holds for $n+1$. Thus, we write $G^{n+1} = G \cdot G^n = G \cdot H_n$. From Lemma 2.1 and the property of matrix multiplication, we get $G^{n+1} = G \cdot H_n = H_{n+1}$. □

As special cases of Theorem 2.1, we note the following special consequences.

- When $k = 1$, we obtain the following fact

$$G = \begin{bmatrix} V_r & (-1)^{r+1} \\ 1 & 0 \end{bmatrix} \text{ and } H_n = G^n = \begin{bmatrix} U_{r(n+1)} & U_{rn} \\ U_{rn} & U_{r(n-1)} \end{bmatrix}.$$

- If we take $A = 1$ and $r = 1$, we get the well-known fact

$$\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}^n = \begin{bmatrix} F_{n+1} & F_n \\ F_n & F_{n-1} \end{bmatrix}.$$

- When $A = 1$, $r = 3$, and $k = 2$, we write

$$a_{1,m} = (-1)^{\binom{m-1}{2}} \begin{Bmatrix} 3 \\ m \end{Bmatrix}_{U,3}; \ 1 \leq m \leq 3$$

and so

$$G = \begin{bmatrix} \frac{F_9}{F_3} & \frac{F_9}{F_3} & -1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \text{ and } H_n = \begin{bmatrix} \frac{F_{3(n+1)}F_{3(n+2)}}{F_3F_6} & \frac{F_{3n}F_{3(n+2)}}{F_3F_3} & -\frac{F_{3n}F_{3(n+1)}}{F_3F_6} \\ \frac{F_{3n}F_{3(n+1)}}{F_3F_6} & \frac{F_{3(n-1)}F_{3(n+1)}}{F_3F_3} & -\frac{F_{3(n-1)}F_{3n}}{F_3F_6} \\ \frac{F_{3(n-1)}F_{3n}}{F_3F_6} & \frac{F_{3(n-2)}F_{3n}}{F_3F_3} & -\frac{F_{3(n-2)}F_{3(n-1)}}{F_3F_6} \end{bmatrix}.$$

Since G is a companion matrix, we derive a linear recursion for the products of general Fibonomial coefficients by the next result.

Theorem 2.2. *For $n, k > 0$, the general Fibonomial coefficients satisfy the recursion*

$$\begin{Bmatrix} n+k+1 \\ k \end{Bmatrix}_{U,r} = \sum_{t=1}^{k+1} s(t) \begin{Bmatrix} k+1 \\ t \end{Bmatrix}_{U,r} \begin{Bmatrix} n+k+1-t \\ k \end{Bmatrix}_{U,r},$$

where the sign function is defined as before.

Proof. By equating $(1,1)$ entries in the equation $H_{n+1} = H_1 H_n$, we obtain $a_{n+1,1} = \sum_{t=1}^{k+1} a_{1,t} a_{n+1-t,1}$. By using the definition of $a_{n,i}$, the proof is obtained after some simplifications. □

Considering the general Fibonomial matrix G, we obtain the following result.

Corollary 2.1. *For $n, r, p > 0$, the following identities hold:*

(i)
$$\begin{Bmatrix} m+n+k+1 \\ k+1-j \end{Bmatrix}_{U,r} \begin{Bmatrix} m+n+j-1 \\ j-1 \end{Bmatrix}_{U,r} = \sum_{t=1}^{k} s(t) \begin{Bmatrix} n+k+1 \\ k+1-t \end{Bmatrix}_{U,r} \begin{Bmatrix} n+t-1 \\ t-1 \end{Bmatrix}_{U,r} \times \begin{Bmatrix} m+k+1-t \\ k-j+1 \end{Bmatrix}_{U,r} \begin{Bmatrix} m-t+j-1 \\ j-1 \end{Bmatrix}_{U,r},$$

(ii)
$$s(i) \begin{Bmatrix} n+1+k \\ n+i \end{Bmatrix}_{U,r} \begin{Bmatrix} n+i-1 \\ n \end{Bmatrix}_{U,r} = s(i)\, s(1) \begin{Bmatrix} k+1 \\ i \end{Bmatrix}_{U,r} \begin{Bmatrix} n+k \\ k \end{Bmatrix}_{U,r} + s(i+1) \begin{Bmatrix} n+k \\ k-i \end{Bmatrix}_{U,r} \begin{Bmatrix} n+i-1 \\ i \end{Bmatrix}_{U,r},$$

where the sign function $s(t)$ is defined as before.

Proof. From the matrix multiplication, we write the equalities $H_{n+1} = H_n H_1$ and $H_{n+m} = H_n H_m$. If we consider these equalities with their corresponding entries, we derive $a_{m+n+1-i,j} = \sum_{t=1}^{k} a_{n+1-i,t} a_{m+1-t,j}$ and $a_{n+1,i} = a_{1,i} a_{n,1} + a_{n,i+1}$. By considering the definition of $a_{n,i}$ and $H = [h_{ij}]$, we obtain the claimed results. □

3. The spectrum of the matrix G

We explicitly find the spectrum of matrix G. Here G is a companion matrix and we recall the fact that for the companion matrix

$$C=\begin{bmatrix} c_1 & c_2 & c_3 & \cdots & c_k \\ 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & 1 & 0 \end{bmatrix},$$

it is known that the $(k+1)$th degree characteristic polynomial of C is

$$\mathcal{C}(x)=x^{k+1}-c_1x^k-c_2x^{k-1}-\ldots-c_{k-1}x-c_k.$$

Denote the characteristic polynomial of the general Fibonomial matrix G of order-k by $\mathcal{G}(x)$. By considering the fact just above and recalling that the first row entries of the companion matrix G are $[G]_{1,m}=a_{1,m}$ with

$$a_{1,m}=s(m)\begin{Bmatrix} k+1 \\ m \end{Bmatrix}_{U,r}$$

for $1\le m\le k$, we have the next result without proof.

Corollary 3.1. *For $n>0$,*

$$\mathcal{G}(x)=\sum_{t=0}^{k+1}s(t)\begin{Bmatrix} k+1 \\ t \end{Bmatrix}_{U,r}x^{k+1-t},$$

where the function $s(t)$ is defined as before.

By the definition of the sign function, for odd r,

$$\mathcal{G}(x)=\sum_{t=0}^{k+1}\begin{Bmatrix} k+1 \\ t \end{Bmatrix}_{U,r}(-1)^{\binom{t+1}{2}}x^{k+1-t}$$

and for even $r>0$,

$$\mathcal{G}(x)=\sum_{t=0}^{k+1}\begin{Bmatrix} k+1 \\ t \end{Bmatrix}_{U,r}(-1)^{t}x^{k+1-t}.$$

In [3,5–7], it was shown that the kth power of generalized Fibonacci numbers $\{U_{nr}\}$ with indices in arithmetic progressions satisfies the following polynomial for $r>0$:

$$\mathcal{C}_n(x)=\sum_{t=0}^{k+1}\begin{Bmatrix} k+1 \\ t \end{Bmatrix}_{U,r}(-1)^{\binom{t+1}{2}}x^{k+1-t};$$

that is, we have that

$$\sum_{t=0}^{k+1}\begin{Bmatrix} k+1 \\ t \end{Bmatrix}_{U,r}(-1)^{\binom{t+1}{2}}U^k_{r(n+k+1-t)}=0.$$

The authors of [1,3] showed that the characteristic polynomial $\mathcal{P}(x)$ of the right-adjusted Pascal matrix $P_n(V_1)=P_n(A)$ for $A=1$ is also equal to the polynomial $\mathcal{C}(x)$ and so $\mathcal{C}(x)=\mathcal{G}(x)=\mathcal{P}(x)$. Thus, we collect these results and derive the following remark:

- the characteristic polynomial $\mathcal{P}(x)$ of the order-k general right-adjusted Pascal matrix $P_k(V_r)$ for $A=1$,
- the characteristic polynomial $\mathcal{G}(x)$ of the order-k general Fibonomial matrix for odd r, and
- the auxiliary polynomial satisfied by the kth power of generalized Fibonacci numbers $\{U_{nr}\}$

are the same.

From [3], we have the next result.

Corollary 3.2. *The roots of the polynomial $\mathcal{P}(x)$ of the general right-adjusted Pascal matrix $P(A)$ of order-k are given by*

$$\begin{array}{ll}\left\{(-1)^j\alpha^{k-1-2j},\ (-1)^j\beta^{k-1-2j}\right\}_{0\le j\le t-1} & \text{if } k=2t,\\ \left\{(-1)^t,\ (-1)^j\alpha^{k-1-2j},\ (-1)^j\beta^{k-1-2j}\right\}_{0\le j\le t-1} & \text{if } k=2t+1,\end{array}$$

where $\alpha,\beta=\left(A\pm\sqrt{A^2+4}\right)/2$.

By considering the polynomial equality $\mathcal{G}(x) = \mathcal{P}(x)$ and Corollary 3.2, we have the eigenvalues of the Fibonomial matrix G as the roots of $\mathcal{P}(x)$.

Corollary 3.3. *The eigenvalues of the Fibonomial matrix G of order-k are given by*

$$\begin{array}{ll} \left\{(-1)^{jr}\alpha^{r(k-1-2j)},\ (-1)^{jr}\beta^{r(k-1-2j)}\right\}_{0\le j\le t-1} & \text{if } k=2t, \\ \left\{(-1)^{tr},\ (-1)^{jr}\alpha^{r(k-1-2j)},\ (-1)^{jr}\beta^{r(k-1-2j)}\right\}_{0\le j\le t-1} & \text{if } k=2t+1, \end{array}$$

where $\alpha,\beta = \left(A \pm \sqrt{A^2+4}\right)/2$.

For example, for $k=3$ and $r=2$, we consider the general Fibonomial matrix G of order 4. Thus,

$$G = \begin{bmatrix} \frac{U_8}{U_2} & -\frac{U_6U_8}{U_2U_4} & \frac{U_8}{U_2} & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \text{ and so } H_n = G^n,$$

where $H_n = [h_{ij}]$ with

$$h_{ij} = (-1)^{j+1} \left\{ {n+4-i \atop 4-j} \right\}_{U,2} \left\{ {n-i+j-1 \atop j-1} \right\}_{U,2} \ :\ 1 \le i,j \le 4.$$

The characteristic polynomial of the matrix G is

$$\mathcal{G}(x) = \sum_{t=0}^{4} (-1)^t \left\{ {4 \atop t} \right\}_{U,2} x^{4-t}$$

and its roots are $\alpha^8, \alpha^4, 1, \beta^4, \beta^8$ where $\alpha,\beta = \left(A \pm \sqrt{A^2+4}\right)/2$.

Corollary 3.4. *Denote the eigenvalues of the Fibonomial matrix G of order-k by μ_i for $1 \le i \le k$. Then*

$$\prod_{i=1}^{k} (x-\mu_i) = \sum_{t=0}^{k} s(t) \left\{ {k \atop t} \right\}_{U,r} x^{k-t}.$$

In [2], the authors showed that $\mathrm{tr}\,(P_k(V_1)) = \frac{U_{kn}}{U_n}$, where $P_n(A)$ is the general Pascal matrix. Generalizing the result of [2], the authors of [12] proved that

$$\mathrm{tr}\,(P_k^n(V_1)) = \frac{U_{rkn}}{U_k}.$$

Since the matrices H_n and $P_k^n(V_r)$ of order-k have the same eigenvalues, we obtain

$$\mathrm{tr}\,(H_n) = \frac{U_{rkn}}{U_r}.$$

Since all eigenvalues of H_n are determined, we easily get the next result.

Theorem 3.1. *For $n > 0$,*

$$\mathrm{tr}\,(H_n) = \sum_{i=0}^{\lfloor k-1/2 \rfloor} (-1)^{inr} V_{(k-2i)nr} + \frac{1}{2}\left(1+(-1)^k\right).$$

4. Diagonalization of G and the Binet formula

In this section, we diagonalize the general Fibonomial matrix G and then derive the Binet formula for the general Fibonomial coefficients $\left\{ {n \atop k} \right\}_{U,r}$. By the definitions, since the eigenvalues $\mu_1, \mu_2, \ldots, \mu_k$ of G of order-k are distinct from each other, the matrix G can be diagonalized.

Define order-k Vandermonde matrix V and order-k diagonal matrix $D = diag(\mu_1, \mu_2, \ldots, \mu_k)$ as

$$V = \begin{bmatrix} \mu_1^k & \mu_2^k & \cdots & \mu_k^k \\ \vdots & \vdots & & \vdots \\ \mu_1^2 & \mu_2^2 & \ldots & \mu_k^2 \\ \mu_1 & \mu_2 & \ldots & \mu_k \\ 1 & 1 & \ldots & 1 \end{bmatrix} \text{ and } D = \begin{bmatrix} \mu_1 & & & \\ & \mu_2 & & \\ & & \ddots & \\ & & & \mu_k \end{bmatrix}.$$

Since $\mu_i \neq \mu_j$ for $1 \le i,j \le k$, $\det V \neq 0$. Let $V_j^{(i)}$ be the matrix of order-k obtained from V^T by replacing the jth column of V by w_i where

$$w_i = \begin{bmatrix} \mu_1^{n-i+k+1} & \mu_2^{n-i+k+1} & \ldots & \mu_k^{n-i+k+1} \end{bmatrix}^T.$$

We give the Binet formula for the generalized Fibonomial coefficients.

Theorem 4.1. *For* $n > 0$,

$$a_{n-i+1,j} = \frac{\det\left(V_j^{(i)}\right)}{\det(V)}.$$

Proof. Since all eigenvalues of the matrix G are different from each other, it can be diagonalized. We write $V^{-1}GV = D$ or $GV = VD$ and so $G^n V = VD^n$. Since V is an invertible matrix and $G^n = H_n = [h_{ij}]$, we write $G^n V = H_{n,k}V = VD^n$. Also, we obtain

$$\begin{aligned} h_{i1}\mu_1^k + h_{i2}\mu_1^{k-1} + \ldots + h_{i,k-2}\mu_1^2 + h_{i,k}\mu_1 + h_{i,k} &= \mu_1^{n-i+k+1} \\ h_{i1}\mu_2^k + h_{i2}\mu_2^{k-1} + \ldots + h_{i,k-2}\mu_2^2 + h_{i,k}\mu_2 + h_{i,k} &= \mu_2^{n-i+k+1} \\ &\vdots \\ h_{i1}\mu_k^k + h_{i2}\mu_k^{k-1} + \ldots + h_{i,k-2}\mu_k^2 + h_{i,k}\mu_k + h_{i,k} &= \mu_k^{n-i+k+1}. \end{aligned}$$

Thus, by Cramer's rule, we have the solution $h_{i,j} = \det\left(V_j^{(i)}\right) / \det(V)$. □

Let $V_j^{(e_i)}$ be a $k \times k$ matrix obtained from the Vandermonde matrix V by replacing the jth column of V by e_i where V is defined as before and e_i is the ith element of the natural basis for $\mathbb{R}^n$ and

$$V_j^{(e_i)} = \begin{bmatrix} \mu_1^k & \cdots & \mu_{j-1}^k & 0 & \mu_{j+1}^k & \cdots & \mu_k^k \\ \vdots & & \vdots & \vdots & \vdots & & \vdots \\ \mu_1^{k-i+1} & \cdots & \mu_{j-1}^{k-i+1} & 0 & \mu_{j+1}^{k-i+1} & \cdots & \mu_k^{k-i+1} \\ \mu_1^{k-i} & \cdots & \mu_{j-1}^{k-i} & 1 & \mu_{j+1}^{k-i} & \cdots & \mu_k^{k-i} \\ \mu_1^{k-i-1} & \cdots & \mu_{j-1}^{k-i-1} & 0 & \mu_{j+1}^{k-i-1} & \cdots & \mu_k^{k-i-1} \\ \vdots & & \vdots & \vdots & \vdots & & \vdots \\ \mu_1 & \cdots & \mu_{j-1} & 0 & \mu_{j+1} & \cdots & \mu_k \\ 1 & \cdots & 1 & \underset{\substack{\downarrow \\ e_i}}{0} & 1 & \cdots & 1 \end{bmatrix}.$$

Let $q_j^{(i)} = \left|V_j^{(e_i)}\right| / |V|$ where the $k \times k$ matrices $V_j^{(e_i)}$ and V are defined as before.

Theorem 4.2. *For* $1 \le n, m \le k$, *it holds that* $a_{n,m} = q_t^{(m)} \mu_t^{n+k}$, *where* $\mu_1, \mu_2, \ldots, \mu_k$ *are the eigenvalues of the matrix* G.

Proof. Consider the Cramer's rule solution of the system

$$\begin{bmatrix} \mu_1^k & \mu_2^k & \cdots & \mu_k^k \\ \vdots & \vdots & & \vdots \\ \mu_1^{k-i+1} & \mu_2^{k-i+1} & \cdots & \mu_k^{k-i+1} \\ \mu_1^{k-i} & \mu_2^{k-i} & \cdots & \mu_k^{k-i} \\ \mu_1^{k-i-1} & \mu_2^{k-i-1} & \cdots & \mu_k^{k-i-1} \\ \vdots & \vdots & & \vdots \\ \mu_1 & \mu_2 & \cdots & \mu_k \\ 1 & 1 & \cdots & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ \vdots \\ x_j \\ \vdots \\ x_{k-1} \\ x_k \end{bmatrix} = \underbrace{\begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \\ 0 \end{bmatrix}}_{e_i},$$

then we obtain

$$q_j^{(i)} = \frac{\left|V_j^{(e_i)}\right|}{|V|} \ (i = 1, 2, \ldots, k).$$

Thus, for $n, k > 0$ and $1 \le m \le k$,

$$a_{n,m} = \sum_{j=1}^{k} q_j^{(m)} \mu_j^{n+k},$$

which completes the proof. □

For example, when $A = 1$, we get $\alpha, \beta = \left(1 \pm \sqrt{5}\right)/2$ and if we choose $k = 2$ and $r = 3$, then

$$G = \begin{bmatrix} \frac{F_9}{F_3} & \frac{F_9}{F_3} & -1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \text{ and } H_n = \begin{bmatrix} \frac{F_{3(n+1)}F_{3(n+2)}}{F_3F_6} & \frac{F_{3n}F_{3(n+2)}}{F_3F_3} & -\frac{F_{3n}F_{3(n+1)}}{F_3F_6} \\ \frac{F_{3n}F_{3(n+1)}}{F_3F_6} & \frac{F_{3(n-1)}F_{3(n+1)}}{F_3F_3} & -\frac{F_{3(n-1)}F_{3n}}{F_3F_6} \\ \frac{F_{3(n-1)}F_{3n}}{F_3F_6} & \frac{F_{3(n-2)}F_{3n}}{F_3F_3} & -\frac{F_{3(n-2)}F_{3(n-1)}}{F_3F_6} \end{bmatrix}.$$

Thus, by Theorem 4.2, we find equivalent statements for the entries of the general Fibonomial matrix $G^n = H_n$. Denote the eigenvalues of the general Fibonomial matrix G, $x^3 - \frac{F_9}{F_3}x^2 - \frac{F_9}{F_3}x + 1 = 0$, by $\gamma_1 = \alpha^6, \gamma_2 = \beta^6, \gamma_3 = -1$. After some computations, we obtain

$$\begin{aligned} q_1^{(1)} &= \frac{1}{(\gamma_1-\gamma_3)(\gamma_1-\gamma_2)}, & q_2^{(1)} &= \frac{1}{(\gamma_2-\gamma_3)(\gamma_2-\gamma_1)}, & q_3^{(1)} &= \frac{1}{(\gamma_2-\gamma_3)(\gamma_1-\gamma_3)}, \\ q_1^{(2)} &= -\frac{\gamma_2+\gamma_3}{(\gamma_1-\gamma_2)(\gamma_1-\gamma_3)}, & q_2^{(2)} &= \frac{\gamma_1+\gamma_3}{(\gamma_2-\gamma_3)(\gamma_1-\gamma_2)}, & q_3^{(2)} &= -\frac{\gamma_1+\gamma_2}{(\gamma_2-\gamma_3)(\gamma_1-\gamma_3)}, \\ q_1^{(3)} &= \frac{\gamma_2\gamma_3}{(\gamma_1-\gamma_3)(\gamma_1-\gamma_2)}, & q_2^{(3)} &= -\frac{\gamma_1\gamma_3}{(\gamma_1-\gamma_2)(\gamma_2-\gamma_3)}, & q_3^{(3)} &= \frac{\gamma_1\gamma_2}{(\gamma_2-\gamma_3)(\gamma_1-\gamma_3)}. \end{aligned}$$

Therefore, by Theorem 4.2 and some arrangements, we obtain

$$F_{3(n+1)}F_{3(n+2)} = \frac{F_{6n+12} + F_{6n+6} + F_6\left(-1\right)^n}{10} \text{ and } F_{3n}F_{3(n+2)} = \frac{F_{6(n+2)} - F_{6n} - F_{12}\left(-1\right)^n}{40}.$$

5. Sums of Fibonomial coefficients

We formulate the sum of the general Fibonomial coefficients with indices in arithmetic progressions of the form

$$S_n = \sum_{i=0}^{n-1} \begin{Bmatrix} k+i \\ k \end{Bmatrix}_{U,r}$$

via matrix methods by extending G which is given in (5). Define the extended matrices T and W_n of order-$(k+2)$ as

$$T = \begin{bmatrix} 1 & 0 & \dots & 0 \\ 1 & & & \\ 0 & & G & \\ \vdots & & & \\ 0 & & & \end{bmatrix} \text{ and } W_n = \begin{bmatrix} 1 & 0 & \dots & 0 \\ S_n & & & \\ \vdots & & H_n & \\ S_{n-k+1} & & & \end{bmatrix},$$

where the sum S_n and the matrices G and H_n are defined as before. By the definition of the matrix $H_n = [h_{i,j}]$, we note that S_n satisfies

$$S_n = \sum_{i=0}^{n-1} \begin{Bmatrix} k+i \\ k \end{Bmatrix}_{U,r} = \sum_{i=0}^{n-1} a_{i,1},$$

where $a_{i,1} = h_{n-i+1,1}$. Then we have the next result.

Theorem 5.1. *For $n > 0$, it holds that $T^n = W_n$.*

Proof. Because of the fact that $S_{n+1} = a_{n,1} + S_n$ and Theorem 2.1, we derive matrix-recurrence relation $W_n = W_{n-1}T$. By the induction method, we write $W_n = W_1T^{n-1}$. By the definition of W_n, we obtain $W_1 = T$ and so $W_n = T^n$. □

From Corollary 3.3, we know that the Fibonomial matrix G has the eigenvalue 1 for both even r and $k \equiv 0 \pmod 2$. Expanding the $\det(\mu I_{k+1} - T)$ with respect to the first row, we see that the matrix T has also the eigenvalue 1. Thus, the matrix T has double eigenvalue 1 for even r and $k \equiv 0 \pmod 2$. For odd r and $k \not\equiv 0 \pmod 2$, we cannot diagonalize the matrix T as it has a double eigenvalue, or in general, the matrix T does not have a linear independent eigenvector associated with the double eigenvalue 1. So, we could not derive an explicit formula for the sum S_n.

Define the matrix M of order $k+2$ as

$$M = \begin{bmatrix} 1 & 0 & \dots & 0 \\ \delta & & & \\ \vdots & & V & \\ \delta & & & \end{bmatrix}$$

where

$$\delta = \left(1 - \sum_{i=1}^{k+1} a_{1,i}\right)^{-1}$$

and the Vandermonde matrix V is defined as before.

It is observed that $TM = MD_1$, where T is as before and $D_1 = diag\,(1, \mu_1, \mu_2, \ldots, \mu_k)$. By the Vandermonde matrix V, computing $\det M$ with respect to the first row shows $\det M = \det V$.

Theorem 5.2. *For even r and $k > 0$ and for $n > 0$, it holds that*

$$S_n = \frac{a_{n,1} + a_{n,2} + \cdots + a_{n,k+1} - 1}{\sum_{i=1}^{k+1} a_{1,i} - 1}.$$

Proof. Since M is invertible, it holds that $M^{-1}TM = D_1$; that is, T is similar to D_1. Thus, we write $T^n M = MD_1^n$. By Theorem 5.1, we have $W_n M = MD_1^n$. By equating the $(2,1)$-th elements of $W_n M = MD_1^n$ and doing a matrix multiplication, we obtain the desired result. □

References

[1] L. Carlitz, The characteristic polynomial of a certain matrix of binomial coefficients, *Fibonacci Quart.* **3(2)** (1965) 81–89.
[2] C. Cooper, R. Kennedy, Proof of a result by Jarden by generalizing a proof by Carlitz, *Fibonacci Quart.* **33(4)** (1995) 304–310.
[3] P. Duvall, T. Vaughan, Pell polynomials and a conjecture of Mahon and Horadam, *Fibonacci Quart.* **26(4)** (1988) 344–353.
[4] H. W. Gould, The bracket function and Fountené–Ward generalized binomial coefficients with application to Fibonomial coefficients, *Fibonacci Quart.* **7** (1969) 23–40.
[5] A. P. Hillman, V. E. Hoggatt, The characteristic polynomial of the generalized shift matrix, *Fibonacci Quart.* **3(2)** (1965) 91–94.
[6] V. E. Hoggatt, Fibonacci numbers and generalized binomial coefficients, *Fibonacci Quart.* **5** (1967) 383–400.
[7] D. Jarden, Recurring sequences, *Riveon Lematematika* **12** (1958) 42–45.
[8] D. Jarden, T. Motzkin, The product of sequences with a common linear recursion formula of order 2, *Riveon Lematematika* **3** (1949) 25–27.
[9] D. Kalman, Generalized Fibonacci numbers by matrix methods, *Fibonacci Quart.* **20(1)** (1982) 73–76.
[10] E. Kılıç, The generalized order-k Fibonacci-Pell sequence by matrix methods, *J. Comput. Appl. Math.* **209(2)** (2007) 133–145.
[11] E. Kılıç, The generalized Fibonomial matrix, *European J. Combin.* **31(1)** (2010) 193–209.
[12] E. Kılıç, G. N. Stanica, P. Stanica, Spectral properties of some combinatorial matrices, *13th International Conference on Fibonacci Numbers and Their Applications*, 2008.
[13] E. Krot, An introduction to finite fibonomial calculus, *Cent. Eur. J. Math.* **2(5)** (2004) 754–766.
[14] N. Li, W. Chu, q-derivative operator proof for a conjecture of Melham, *Discrete Appl. Math.* **177** (2014) 158–164.
[15] D. Marques, P. Trojovsky, On some new identities for the Fibonomial coefficients, *Math. Slovaca* **64(4)** (2014) 809–818.
[16] J. Seibert, P. Trojovsky, On some identities for the Fibonomial coefficients, *Math. Slovaca* **55** (2005) 9–19.
[17] P. Stanica, Netted matrices, *Int. J. Math. Math. Sci.* **39** (2003) 2507–2518.
[18] R. F. Torretto, J. A. Fuchs, Generalized binomial coefficients, *Fibonacci Quart.* **2** (1964) 296–302.
[19] P. Trojovsky, On some identities for the Fibonomial coefficients via generating function, *Discrete Appl. Math.* **155(15)** (2007) 2017–2024.
[20] S. Vajda, *Fibonacci* & *Lucas Numbers, and the Golden Section*, Wiley, New York, 1989.