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Research Article

On set systems without singleton intersections

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Abstract

Consider a family $\mathcal{F}$ of k-subsets of an ambient (k^2-k+1)-set such that no pair of k-subsets in $\mathcal{F}$ intersects in exactly one element. In this article, it is shown that the maximum size of $\mathcal{F}$ is $\binom{k^2-k-1}{k-2}$ for every $k>1$.

Keywords: Johnson scheme; Erdős–Sós problem; finite projective planes.

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1. Introduction

A large branch of combinatorics grows from the celebrated Erdős–Ko–Rado theorem [7], which states that a family of pairwise intersecting k-element subsets of an n-element set has size at most $\binom{n-1}{k-1}$ provided that $n \geq 2k$.

For our aim, it is convenient to define a *Johnson graph* $J(n,k,t)$, whose vertices are k-element subsets of an n-element set and edges connect pairs of vertices with intersection t. An *independent set* is a vertex subset of a graph such that there is no edge between its elements. Let $\alpha(G)$ stand for the size of a maximal independent set in a graph G. In this language, the statement of the Erdős–Ko–Rado theorem is

$$\alpha(J[n,k,0]) = \binom{n-1}{k-1}$$

for $n \geq 2k$. The problem of finding $\alpha(J[n,k,t])$ is known as Erdős–Sós forbidden intersection problem. The bibliography on this problem is wide, and the proofs use very different techniques. Let us briefly provide the highlights. Frankl and Füredi [9] used the so-called Δ-system method to show that

$$\alpha(J[n,k,t]) = \binom{n-t}{k-t}$$

for $n > n_0(k)$ and $k \geq 2t+2$. Another very general result was obtained by Frankl and Wilson [10] by a rank bound: it gives

$$\alpha(J[n,k,t]) \leq \binom{n}{k-t-1}$$

for $k > 2t$ and $k-t$ being a prime power. This result has important applications to discrete geometry (see [13, 19]).

Recently, Ellis, Keller, and Lifshitz [6, 15] used the junta-method to determine $\alpha(J[n,k,t])$ for $\varepsilon < k/n < 1/2-\varepsilon$ and $n > n_0(t,\varepsilon)$. Kupavskii and Zakharov [16] showed that

$$\alpha(J[n,k,t]) = \binom{n-t-1}{k-t-1}$$

for $k > k_0$, $n = \lceil k^\alpha \rceil$, $t = \lceil k^\beta \rceil$, where $\alpha > 1$ and $1/2 > \beta > 0$ satisfy $\alpha > 1+2\beta$. They use the spread approximation technique [2].

The problem for $t=1$ and $n > n_0(k)$ was solved by Frankl [8]. Also, we mention here that $\alpha(J[n,4,1]) = \binom{n-2}{2}$ for $n \geq 9$ (see [14]). The main result of the present paper is stated as follows.

Theorem 1.1. *For every $k>1$,*

$$\alpha(J[k^2-k+1,k,1]) = \binom{k^2-k-1}{k-2}.$$

Note that for the case where $k > k_0$, Theorem 1.1 follows from the mentioned results of Kupavskii and Zakharov [16] and Keller and Lifshitz [15].

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2. Tools

Johnson scheme and Bose–Mesner algebra

The facts from this subsection can be found in books [3, 11].

An associative scheme is a pair $(V, \mathcal{R})$ consisting of a finite set V and a family of non-empty binary relations $\mathcal{R}$ on V, and satisfying the following properties:

1. sets $R \in \mathcal{R}$ form a partition of V^2;
2. the diagonal $\Delta(V)$ of the set V^2 is an element of $\mathcal{R}$;
3. the set $\mathcal{R}$ is closed under the interchange of the first and second coordinates in V^2;
4. for arbitrary relations $R, S, T \in \mathcal{R}$ numbers
$$|v \in V : (u,v) \in R, (v,w) \in S|$$
are the same for all $(u,w) \in T$.

The classical *Johnson scheme* is given by

$$V = \binom{[n]}{k}; \quad R_i := \{(v,u) \in V \times V : \langle v,u \rangle = k - i\}, \quad i = 0, \dots k.$$

The Bose–Mesner algebra of the Johnson scheme is the algebra of $\binom{n}{k} \times \binom{n}{k}$ matrices, with entries defined by

$$A(x,y) := f(|x \cap y|).$$

Since this is indeed a commutative algebra and all such matrices are symmetric, these matrices are simultaneously diagonalizable. A standard basis of the Bose–Mesner algebra is formed by matrices

$$B_i(x,y) := \binom{|x \setminus y|}{i}, \quad i = 0, \dots, k.$$

The eigenvalues of B_i are given by

$$\mu_j^{(i)} = (-1)^j \binom{k-j}{i-j} \binom{n-i-j}{k-j}.$$

Then the machinery offers to represent any matrix A from the Bose–Mesner algebra as $\sum b_i B_i$ and get its spectrum as

$$\lambda_j = \sum_{i=0}^{k} b_i \mu_j^{(i)}.$$

Let I be a subset of $\binom{[n]}{k}$ and χ_I stand for its characteristic vector. Denote by c_i, $i = 0, \dots, k$ the coefficients in the decomposition of χ_I with respect to the common eigenspaces of the Bose–Mesner algebra.

Hoffman bound

The following celebrated theorem is widely known as the Hoffman ratio bound [12].

Theorem 2.1. *Let A be a pseudo-adjacency matrix of a d-regular N-vertex graph G with non-negative entries. Then*

$$\alpha(G) \leq N \frac{-\lambda_{min}}{d - \lambda_{min}}, \tag{1}$$

where $\alpha(G)$ is the independence number of G.

The proof is a one-line collection of several simple observations. Let I be any independent set in G and χ_I stand for its characteristic vector. Then

$$0 = (A\chi_I, \chi_I) = \sum_{i=1}^{N} a_i^2 \lambda_i \geq a_1^2 d + (a_2^2 + \cdots + a_N^2)\lambda_{min} = \frac{|I|^2}{N} d + \left(|I| - \frac{|I|^2}{N}\right) \lambda_{min}, \tag{2}$$

where a_i are the coefficients in the decomposition of χ_I in the eigenbasis of A. Here we use that a spectral radius of a d-regular graph is d and it is achieved at the all-unit vector. Also, in the case of an edge-transitive graph, the Hoffman bound coincides with Lovász theta-bound [18].

3. Proof of Theorem 1.1

An example of an independent set of size $\binom{k^2-k-1}{k-2}$ is given by a collection of all sets, containing elements 1 and 2. Clearly, the adjacency matrix of the Johnson graph $J(n,k,1)$ belongs to the Bose–Mesner algebra with

$$f(1) = 1,\ f(0) = f(2) = \cdots = f(k) = 0.$$

It is straightforward to check that the coefficients in the standard basis of the Bose–Mesner algebra are the following:

$$b_0 = b_1 = \cdots = b_{k-2} = 0, \quad b_{k-1} = 1, \quad b_k = -k$$

and

$$\lambda_0 = -k\binom{n-k}{k} + k\binom{n-k+1}{k} = k\binom{n-k}{k-1},$$

$$\lambda_1 = k\binom{n-k-1}{k-1} - (k-1)\binom{n-k}{k-1}, \quad \lambda_2 = -k\binom{n-k-2}{k-2} + (k-2)\binom{n-k-1}{k-2}.$$

For $n = k^2-k+1$, we have

$$\lambda_1 = \lambda_2 = -\frac{1}{k-1}\binom{n-k}{k-1} < 0$$

and

$$\lambda_3 = k\binom{n-k-3}{k-3} - (k-3)\binom{n-k-2}{k-3} = \frac{2k^2-3k-3}{k^2-3k+2}\binom{n-k-3}{k-3} > 0.$$

Also,

$$|\lambda_4|, |\lambda_5|, \ldots, |\lambda_k| \leq k\binom{n-k}{k-3}.$$

Hence, $\lambda_1 = \lambda_2$ are the smallest eigenvalues.

Now the upper bound follows from the Hoffman bound (1):

$$\frac{-\lambda_{min}}{d-\lambda_{min}} = \frac{1}{k^2-k+1} = \frac{\binom{n-2}{k-2}}{\binom{n}{k}}.$$

4. Discussion

Let us briefly discuss the sporadic nature of the result. In all solved cases, maximal independent sets form designs or juntas. The Hoffman bound is tight when the corresponding characteristic vector belongs to maximal and minimal eigenspaces, and it seems difficult to modify the method in other cases. The maximal eigenspace is always unique and corresponds to the all-unit vector. For $t \geq 1$, the characteristic vectors of all known examples belong to more than two eigenspaces, so several minimal eigenvalues should coincide in order to use the Hoffman bound. Summing up, it seems that the only case is $t = 1$, in which we have an example with the characteristic vector in the first three eigenspaces. So we need $\lambda_1 = \lambda_2$ which implies $n = k^2 - k + 1$.

Finite projective planes

If $k-1$ is a prime power, then one can prove the upper bound in Theorem 1.1 combinatorially (see [4]). Since Johnson graphs are vertex-transitive (moreover, they are edge-transitive), one has

$$w(J) \cdot \alpha(J) \leq |V(J)|.$$

In our case, k^2-k+1 is the size of a projective plane over $GF(k-1)$, and so $w(J[k^2-k+1]) \geq k^2-k+1$. This immediately implies the bound.

However, for a composite $k-1$, the corresponding construction may not exist. A major negative result is a celebrated Bruck–Ryser theorem [5] which states that if n is a positive integer of the form $4k+1$ or $4k+2$ and n is not equal to the sum of two integer squares, then n does not occur as the order of a finite plane. A widely known conjecture is that the order of a finite plane is always a prime power. Also, the non-existence of a finite plane of order 10 was proven by Lam [17].

Uniqueness

For $k = 3$, the graph $J(7,3,1)$ has a maximal independent set of another structure, namely

$$\{\{1,2,3\},\{1,2,4\},\{1,3,4\},\{2,3,4\},\{5,6,7\}\}.$$

For other values of k, it seems very likely that every maximal independent set of $J(k^2-k+1,k,1)$ forms a family with two elements in common (it was also conjectured by Aljohani, Bamberg, and Cameron [1]). However, we are not able to prove it. Following the proof of Theorem 1.1, an independent set I of the maximal size satisfies the equality in (2), and thus χ_I belongs to the zeroth, the first and the second eigenspaces. The main obstacle in our attempt is a relatively complicated structure of the second eigenspace.

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