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Research Article

Parking functions and Łukasiewicz paths

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Abstract

We present a bijection between two well-known objects in the ubiquitous Catalan family: non-decreasing parking functions and Łukasiewicz paths. This bijection maps the maximum displacement of a parking function to the height of the corresponding Łukasiewicz path, and the total displacement to the area of the path. We also study this bijection restricted to two specific families of parking functions: unit-interval parking functions and prime parking functions.

Keywords: parking functions; Łukasiewicz paths; Catalan numbers; displacement statistic; bijection.

2020 Mathematics Subject Classification: 05A19, 05A05, 05A10.

1. Introduction

The Catalan numbers, defined by $\mathrm{Cat}_n := \frac{1}{n+1}\binom{2n}{n}$, are ubiquitous in combinatorics. The corresponding entry, Sequence A000108 in the OEIS [19], states: "This is probably the longest entry in the OEIS, and rightly so". Stanley's book, *Catalan Numbers* [18], offers hundreds of combinatorial interpretations of these numbers. Let us simply list some of the most famous: Dyck paths, plane trees, non-crossing partitions, permutations avoiding any single pattern of length 3, and so on.

In this paper, we present a bijection between two objects of the Catalan family: non-decreasing parking functions, and Łukasiewicz paths (see Equation (2)). Surprisingly, while this bijection appears implicitly in some works (see e.g. [6, Section 1.4]), and can be obtained by composing various well-known bijections such as those in Stanley's book [18], it does not appear to have been explicitly stated in the literature. In particular, the preservation of various statistics (see below) does not seem to have been previously noted and is one of our main contributions.

Our paper is organised as follows. In the remainder of this section, we introduce parking functions and Łukasiewicz paths, and recall some of their important properties. Section 2 establishes the main result of this paper: a bijection between non-decreasing parking functions and Łukasiewicz paths. We study the effects of the bijection on the *displacement* of parking functions, which measures how far away cars end up from their preferred spots. More precisely, in Theorem 2.1 we will see that the total displacement of a parking function maps to the area of the corresponding Łukasiewicz path, and the maximum displacement to the height. We also present a simple algorithmic procedure to get the inverse bijection from Łukasiewicz paths to non-decreasing parking functions (Algorithm 1 and Theorem 2.2). Finally, in Section 3 we study various specialisations of this bijection to families of parking functions and Łukasiewicz paths with added restrictions.

1.1. Parking functions

Throughout this paper, n denotes a positive integer, and we let $[n] := \{1, \ldots, n\}$. Consider a one-directional car park consisting of n spots labelled 1 to n, and n cars also labelled 1 to n. A *parking preference* is a vector $p = (p_1, \ldots, p_n) \in [n]^n$, with p_i denoting the preferred parking spot of car i for each $i \in [n]$. The cars enter the car park sequentially in order 1 to n. If spot p_i is empty when car i enters, then car i parks in spot p_i. Otherwise, if spot p_i has already been occupied by some previous car $j < i$, car i cannot park in spot p_i. In that case, the car drives on and parks in the first unoccupied spot $k > p_i$. If no such spot exists, car i exits the car park and fails to park. We say that p is a *parking function* if all cars are able to park through this process (see Example 1.1). We denote by PF_n the set of parking functions with n cars/spots.

Definition 1.1. *Given a parking function $p = (p_1, \ldots, p_n)$, the* outcome *of p is the sequence $\mathcal{O}(p) = (o_1, \ldots, o_n)$, where for each car $i \in [n]$, o_i is the spot where car i ends up parking.*

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Definition 1.2. *Given a parking function $p = (p_1, \ldots, p_n) \in \mathrm{PF}_n$, with outcome $\mathcal{O}(p) = (o_1, \ldots, o_n)$, the* displacement d_i *of car i describes the distance between the initial preference p_i of car i and the actual spot where car i ends up, i.e. $d_i = o_i - p_i$. The* displacement vector *of p is then $\mathrm{disp}(p) := (d_1, d_2, \ldots, d_n)$, and the* total displacement *of p is $|\mathrm{disp}(p)| := \sum_{i \in [n]} d_i$.*

Example 1.1. *Consider the parking preference $p = (2, 1, 4, 4, 1)$. We first describe the parking process for p, which is illustrated in Figure 1.1. Initially, car 1 parks in spot 2, followed by car 2 parking in spot 1, and car 3 in spot 4 (none of these spots are occupied when the cars arrive). When car 4 arrives, it wants to park in spot 4. However, spot 4 is occupied by car 3, so car 4 drives on to find the first available spot and park in it, which is spot 5. Finally, car 5 wants to park in spot 1, but spot 1 has been occupied by car 2, causing car 5 to drive on: spot 2 is also occupied by car 1, so car 5 ends up parking in spot 3 (which is the first available spot at this point). Finally, all cars are able to park, so p is a parking function. Moreover, we get the outcome $\mathcal{O}(p) = (2, 1, 4, 5, 3)$, and displacement vector $\mathrm{disp}(p) = (0, 0, 0, 1, 2)$, with total displacement $|\mathrm{disp}(p)| = 3$.*

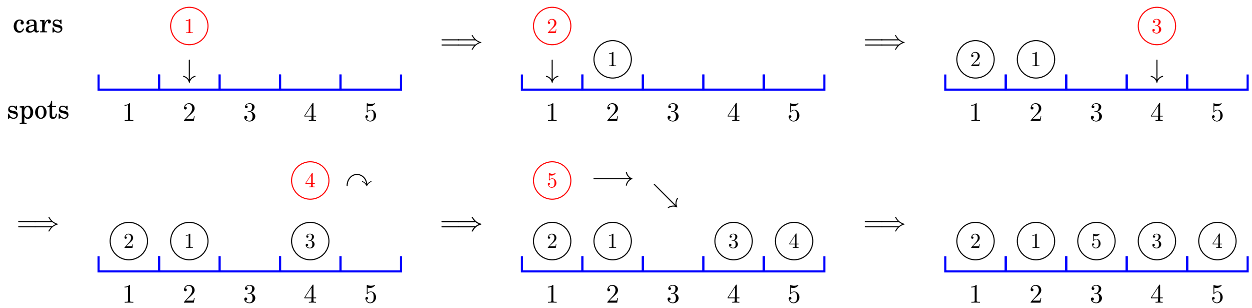

Figure 1.1: The parking process for $p = (2, 1, 4, 4, 1) \in \mathrm{PF}_5$.

Parking functions were originally introduced by Konheim and Weiss [12] to study a hashing technique known as linear probing. Since then, they have been a popular research topic in Mathematics and Computer Science, with rich combinatorial connections to fields such as hyperplane arrangements [16] or statistical physics [5]. We refer the interested reader to the excellent survey by Yan [20]. The following result gives two classical characterisations of parking functions (see [20]).

Proposition 1.1. *Let $p = (p_1, \ldots, p_n) \in [n]^n$ be a parking preference. Define $p^{\mathrm{inc}} = (p_1^{\mathrm{inc}}, \ldots, p_n^{\mathrm{inc}})$ be its non-decreasing re-arrangement. Then the following are equivalent.*

1. *We have $p \in \mathrm{PF}_n$.*
2. *For all $i \in [n]$, we have $p_i^{\mathrm{inc}} \leq i$.*
3. *For all $i \in [n]$, we have $|\{j \in [n];\, p_j \leq i\}| \geq i$.*

In this paper, we will be primarily interested in *non-decreasing* parking functions. These are parking functions which are in weakly increasing order, i.e. $p_i \leq p_{i+1}$ for all $i \in [n-1]$. We denote $\mathrm{PF}_n^{\mathrm{inc}}$ the set of non-decreasing parking functions of length n. There is a classical bijection between non-decreasing parking functions and Dyck paths, which we recall briefly here (see also [20, Page 54]).

For our purposes, a Dyck path will be a lattice path from $(0, 0)$ to (n, n) for some $n \geq 0$ with steps $E = (1, 0)$ and $N = (0, 1)$, which never goes above the diagonal $y = x$ (see Figure 1.2). Note that a Dyck path w is uniquely determined by the weakly increasing sequence $0 = h_1 \leq \cdots \leq h_n \leq n-1$ of heights (y-coordinates) of its E steps. Formally, we define $w = w(h_1, \ldots, h_n) := EN^{h_2-h_1}EN^{h_3-h_2}\cdots EN^{n-h_n}$, where the notation N^k indicates the step N repeated k times.

It is then straightforward to see that the map $p = (p_1, \ldots, p_n) \mapsto w(p_1 - 1, \ldots, p_n - 1)$ is a bijection from the set $\mathrm{PF}_n^{\mathrm{inc}}$ of non-decreasing parking functions of length n to the set of Dyck paths with $2n$ steps. In particular, we have $|\mathrm{PF}_n^{\mathrm{inc}}| = \mathrm{Cat}_n$, the n-th Catalan number. Moreover, for $p \in \mathrm{PF}_n^{\mathrm{inc}}$, the total displacement $|\mathrm{disp}(p)|$ of p is equal to the *area* $\mathrm{Area}(w)$ of the corresponding Dyck path w, defined as the number of complete lattice squares between the path w and the line $y = x$.

Example 1.2. *Consider the non-decreasing parking function $p = (1, 1, 2, 4, 4) \in \mathrm{PF}_5^{\mathrm{inc}}$. The corresponding height sequence is $h = (0, 0, 1, 3, 3)$, yielding the Dyck path $w = EENENNEENN$ as in Figure 1.2 below. Here we label the i-th E step with the value $p_i = h_i + 1$ for each $i \in [n]$. We can check that $\mathrm{disp}(p) = (0, 1, 1, 0, 1)$ (see also Fact 1.1), which gives $|\mathrm{disp}(p)| = 3 = \mathrm{Area}(w)$ (given by the shaded lattice squares).*

Remark 1.1. *Because of this correspondence to the area statistic of Dyck paths, the total displacement statistic is also sometimes referred to as the* area *of parking functions. This statistic has been studied in previous work, including by Kreweras [13] who showed that it is equi-distributed with the inversion statistic on labelled plane trees. More recently, Colmenarejo et al. [4] studied the area statistic on a generalisation of parking functions called k-*Naples *parking functions where, if a car's preferred spot is occupied, it is first allowed to reverse up to k spots before driving on.*

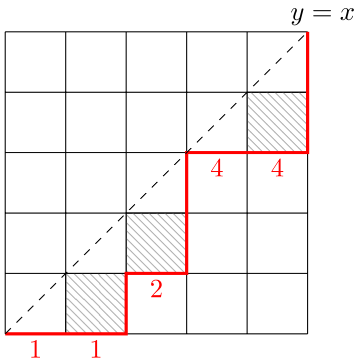

Figure 1.2: The Dyck path corresponding to the non-decreasing parking function $p = (1, 1, 2, 4, 4) \in \mathrm{PF}_5^{\mathrm{inc}}$.

We end this section with the simple following fact, which will prove useful in Section 2.

Fact 1.1. *Let $p \in \mathrm{PF}_n^{\mathrm{inc}}$ be a non-decreasing parking function. Then we have $\mathcal{O}(p) = (1, 2, \ldots, n)$.*

Proof. First, observe that for any parking function $p = (p_1, \ldots, p_n)$, if two cars $i < j$ have preferences $p_i \leq p_j$, and final parking spots o_i and o_j, then we have $o_i < o_j$. Indeed, by definition, o_i is the first spot $k \geq p_i$ which is available when car i enters the car park. In particular, once i has parked, all spots between p_i and o_i (both included) are occupied. Since car j enters after car i, and parks in the first available spot $k' \geq p_j \geq p_i$, this spot must be after o_i, as desired. Now assume that p is non-decreasing. From the previous observation we see that the sequence $\mathcal{O}(p)$ must be increasing, and $(1, 2, \ldots, n)$ is the only increasing sequence of $[n]$ of length n. □

1.2. Łukasiewicz paths

Definition 1.3. *A Łukasiewicz* word *of length n is a sequence $\ell = (\ell_1, \ldots, \ell_n)$ of integers $\ell_i \geq -1$ such that:*

- *For any $k \in [n]$, we have $\sum_{i=1}^{k} \ell_i \geq 0$.*
- *We have $\sum_{i=1}^{n} \ell_i = 0$.*

Łukasiewicz words are usually represented as certain lattice paths, by associating to each ℓ_i the step $(1, \ell_i)$. In this work, we choose a slightly different representation by instead taking steps of the form $s^k := (k+1, k)$. We will refer to k as the *size* of the step s^k. In this setting, a Łukasiewicz path is a lattice path with n steps in the set $S = \{s^k\}_{k \geq -1}$, which starts at $(0, 0)$, ends at $(n, 0)$ and never goes below the x-axis. Figure 1.3 shows an example of a Łukasiewicz path with $n = 12$ steps. There is an obvious bijection between Łukasiewicz words of length n and Łukasiewicz paths with n steps by mapping each element ℓ_i in the word to the step s^{ℓ_i} in the path. With slight abuse of notation, we identify these two sets, denoting them Luk_n, i.e. we write $\ell \in \mathrm{Luk}_n$ to refer to a Łukasiewicz word or path, depending on context.

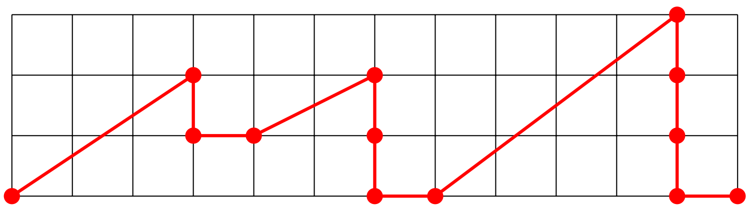

Figure 1.3: Example of a Łukasiewicz path corresponding to the word $\ell = (2, -1, 0, 1, -1, -1, 0, 3, -1, -1, -1, 0)$.

Łukasiewicz paths were named after the Polish mathematician Jan Łukasiewicz. While maybe not as ubiquitous as their lattice path cousins, Dyck paths, and Motzkin paths, they have nonetheless been a rich research topic in combinatorics and discrete probability. Perhaps the most famous use of Łukasiewicz words is as a bijective encoding of rooted plane trees, see e.g. [9, Chapter 1, Section 5]. For this, we map a rooted plane tree T with $n+1$ nodes to the word $\ell = (\ell_1, \ldots, \ell_n)$, where ℓ_i is one less than the number of children of the i-th node visited in the depth-first search (DFS) of T (the last node visited in a DFS is always a leaf, so we omit it in the corresponding word). The bijection implies in particular that $|\mathrm{Luk}_n| = \mathrm{Cat}_n$. Figure 1.4 shows an example of this encoding, with the nodes on the plane tree labelled according to their DFS index.

One statistic of interest for a Łukasiewicz path ℓ is its *height*, denoted $\mathrm{Height}(\ell)$, defined to be the largest y-coordinate reached by the path. For example, the Łukasiewicz path in Figure 1.3 has height 3. Another statistic of interest is the *area*

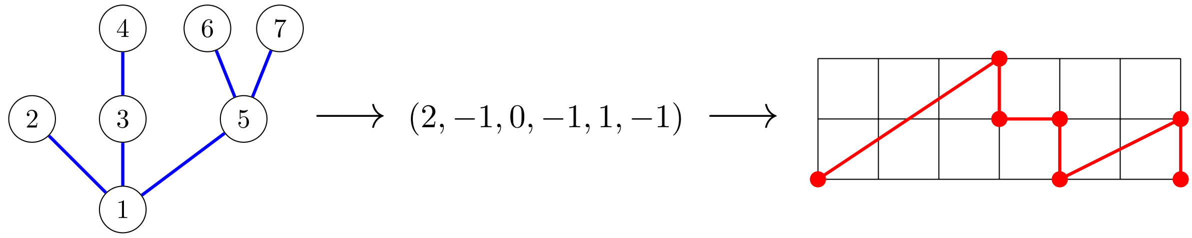

Figure 1.4: A plane tree with nodes labelled according to the DFS (left) together with its Lukasiewicz word encoding (middle) and the corresponding representation as a Lukasiewicz path (right).

of ℓ, defined to be the area between the path and the x-axis. We denote this $\mathrm{Area}\,(\ell)$. This can be computed through the following observation. If ℓ takes a step $s^k = (k+1, k)$ starting at height h above the x-axis (see Figure 1.5), then the area $\mathcal{A}(k, h)$ "under" this step is simply the area of a trapezium, given by

$$\mathcal{A}(k,h) = (k+1)\cdot\frac{h+(h+k)}{2} = \frac{(k+1)(2h+k)}{2}. \tag{1}$$

To obtain the area of the entire path, we simply take the sums of all these areas. For example, for the Łukasiewicz path ℓ in Figure 1.3, we get $\mathrm{Area}\,(\ell) = 3+0+1+2\cdot 3/2+0+0+0+4\cdot 3/2+0+0+0+0 = 13$.

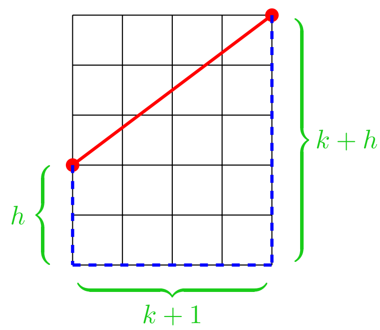

Figure 1.5: A step $s^k = (k+1, k)$ starting at height h. The area under this step is given by Equation (1).

2. The main result

We are now equipped to define the map from parking functions to Łukasiewicz paths. Given a parking function $p = (p_1, \ldots, p_n) \in \mathrm{PF}_n$, we define a sequence $\Psi_{\mathrm{PF}\to\mathrm{Luk}}(p) = \ell = (\ell_1, \ldots, \ell_n)$ by:

$$\forall i \in [n],\ \ell_i := |\{j \in [n];\ p_j = i\}| - 1. \tag{2}$$

In other words, ℓ_i is one less than the number of cars whose preferred spot is i in the parking function p. In particular, we have $\ell_i \geq -1$. The following is then a straightforward consequence of Proposition 1.1 (Characterisation (3)) and of Definition 1.3.

Proposition 2.1. *For any parking function $p \in \mathrm{PF}_n$, we have $\Psi_{\mathrm{PF}\to\mathrm{Luk}}(p) \in \mathrm{Luk}_n$.*

We are now equipped to state the main result of this paper.

Theorem 2.1. *The map $\Psi_{\mathrm{PF}\to\mathrm{Luk}} : \mathrm{PF}_n^{\mathrm{inc}} \to \mathrm{Luk}_n$ is a bijection. Moreover, for any $p \in \mathrm{PF}_n^{\mathrm{inc}}$, we have $|\mathrm{disp}\,(p)| = \mathrm{Area}\,(\Psi_{\mathrm{PF}\to\mathrm{Luk}}(p))$, and $\max\,(\mathrm{disp}\,(p)) = \mathrm{Height}\,(\Psi_{\mathrm{PF}\to\mathrm{Luk}}(p))$.*

Example 2.1. *Consider the non-decreasing parking function $p = (1,1,1,3,4,4,7,8,8,8,8,12) \in \mathrm{PF}_{12}^{\mathrm{inc}}$. For each car i, the displacement d_i is given by $d_i = o_i - p_i = i - p_i$ by Fact 1.1, yielding $\mathrm{disp}\,(p) = (0,1,2,1,1,2,0,0,1,2,3,0)$. In particular, we get total displacement $|\mathrm{disp}\,(p)| = 13$, and maximum displacement $\max\,(\mathrm{disp}\,(p)) = 3$. The corresponding Łukasiewicz word is given by $\ell = \Psi_{\mathrm{PF}\to\mathrm{Luk}}(p) = (2,-1,0,1,-1,-1,0,3,-1,-1,-1,0)$, whose lattice path is exactly the Łukasiewicz path illustrated in Figure 1.3, which has $\mathrm{Area}\,(\ell) = 13$ and $\mathrm{Height}\,(\ell) = 3$, as desired.*

Remark 2.1. *The bijection from non-decreasing parking functions to Dyck paths from Section 1.1 can be similarly defined if we consider a Dyck path to be a lattice path from $(0,0)$ to $(2n,0)$ with steps $U = (1,1)$ and $D = (1,-1)$ which never goes below the x-axis. In this setting, the Dyck path w corresponding to $p \in \mathrm{PF}_n^{\mathrm{inc}}$ is defined by $w = U^{q_1}DU^{q_2}D\cdots U^{q_n}D$, where*

$q_i := |\{j \in [n];\ p_j = i\}|$ *for each* $i \in [n]$. *Compared to this construction, our map has the disadvantage of needing to subtract one from each* q_i *(see Equation (2)). However, the benefits of our map are that we directly get the sequence of steps in the corresponding path (without needing to insert the* D *steps as in the Dyck path), as well as more direct mappings from the total and maximum displacement statistics to the area and height of the path.*

To show that $\Psi_{\mathrm{PF}\to\mathrm{Luk}}$ is a bijection, we exhibit its inverse in Algorithm 1. We will use square brackets instead of standard parentheses to delimit sequences, using the latter instead to indicate precedence order (as on Line 3). Here $\varepsilon := []$ denotes the empty sequence, and for two sequences $a = [a_1, \dots, a_x]$, $b = [b_1, \dots, b_y]$, and an integer m, we write $a + b := [a_1, \dots, a_x, b_1, \dots, b_y]$ for the concatenation of a and b, and $a * m := a + \dots + a$ (repeated m times), with the convention $a * 0 = \varepsilon$. As usual, multiplication takes precedence over addition.

Algorithm 1 From Łukasiewicz words to parking functions

Require: $\ell = [\ell_1, \dots, \ell_n] \in \mathrm{Luk}_n$
1: **Initialise:** $p \leftarrow \varepsilon$
2: **for** $i = 1$ to n **do**
3: $\quad p \leftarrow p + [i] * (\ell_i + 1)$ $\triangleright$ Append i to p $(\ell_i + 1)$ times
4: **end for**
5: **return** $p := \Psi_{\mathrm{Luk}\to\mathrm{PF}}(\ell)$

Theorem 2.2. *For any Łukasiewicz word* $\ell \in \mathrm{Luk}_n$, *Algorithm 1 outputs a non-decreasing parking function* $p = \Psi_{\mathrm{Luk}\to\mathrm{PF}}(\ell) \in \mathrm{PF}_n^{\mathrm{inc}}$. *Moreover, the maps* $\Psi_{\mathrm{PF}\to\mathrm{Luk}} : \mathrm{PF}_n^{\mathrm{inc}} \to \mathrm{Luk}_n$ *and* $\Psi_{\mathrm{Luk}\to\mathrm{PF}} : \mathrm{Luk}_n \to \mathrm{PF}_n^{\mathrm{inc}}$ *are inverses of each other.*

Remark 2.2. *The non-decreasing parking function* $p = (p_1, \dots, p_n) = \Psi_{\mathrm{Luk}\to\mathrm{PF}}(\ell)$ *corresponding to a Łukasiewicz path* $\ell \in \mathrm{Luk}_n$ *can also be read "graphically" as follows. For each* $i \in [n]$, p_i *is the index* j *of the step* ℓ_j *of* ℓ *which crosses the region from* $x = i - 1$ *to* $x = i$. *Figure 2.1 illustrates this construction for the Łukasiewicz path* $\ell = [2, -1, 0, 1, -1, -1, 0, 3, -1, -1, -1, 0]$. *The parking function* $p = \Psi_{\mathrm{Luk}\to\mathrm{PF}}(\ell)$ *is obtained by reading the blue indices below the* x*-axis from left-to-right, yielding* $p = (1, 1, 1, 3, 4, 4, 7, 8, 8, 8, 8, 12)$. *It is straightforward to check that Algorithm 1 gives the same parking function.*

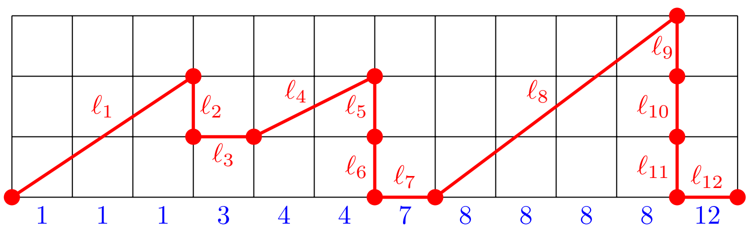

Figure 2.1: Illustrating the construction from Łukasiewicz paths to non-decreasing parking functions.

Proof of Theorem 2.2. By construction, Algorithm 1 outputs a non-decreasing sequence p of elements in $[n]$, whose length is $\sum_{i=1}^{n}(\ell_i + 1) = 0 + n = n$, where we use the fact that the elements of a Łukasiewicz word sum to 0. Therefore p is a parking preference. Now fix some $i \in [n]$. By construction, we have $|\{j \in [n];\ p_j \leq i\}| = \sum_{j=1}^{i}(\ell_j + 1) \geq 0 + i = i$ (again applying Definition 1.3). Therefore p is a parking function by Proposition 1.1. The fact that the maps $\Psi_{\mathrm{PF}\to\mathrm{Luk}} : \mathrm{PF}_n^{\mathrm{inc}} \to \mathrm{Luk}_n$ and $\Psi_{\mathrm{Luk}\to\mathrm{PF}} : \mathrm{Luk}_n \to \mathrm{PF}_n^{\mathrm{inc}}$ are inverses of each other follows immediately from their constructions. □

We now turn to the proof of Theorem 2.1. Theorem 2.2 implies that the map $\Psi_{\mathrm{PF}\to\mathrm{Luk}} : \mathrm{PF}_n^{\mathrm{inc}} \to \mathrm{Luk}_n$ is a bijection, so it remains to show the equalities relating to the displacement statistic. For a Łukasiewicz path $\ell = (\ell_1, \dots, \ell_n) \in \mathrm{Luk}_n$ and an index $j \in [n]$, we denote by $h(\ell; j) := \sum_{i=1}^{j} \ell_i$ the height of the path ℓ after j steps.

Lemma 2.1. *Let* $p \in \mathrm{PF}_n^{\mathrm{inc}}$ *be a non-decreasing parking function, and* $\ell := \Psi_{\mathrm{PF}\to\mathrm{Luk}}(p)$ *the corresponding Łukasiewicz path. For any* $j \in \{0, \dots, n\}$, *we have:*

$$h(\ell; j) = |\{i \in [n];\ p_i \leq j\}| - j. \tag{3}$$

Proof. By definition, for any $i \in [n]$ we have $\ell_i = |\{k \in [n];\ p_k = i\}| - 1$. Equation (3) then follows through summation. □

One may think of the right-hand side of Equation (3) as measuring *excess cars*: it counts the number of cars that wish to park in or before spot j but will be unable to do so (i.e. end up parking in some spot $k > j$). Lemma 2.1 then states that $\Psi_{\mathrm{PF}\to\mathrm{Luk}}$ maps the excess cars statistic to the height of the corresponding Łukasiewicz path.

Proof of Theorem 2.1. Let $p \in \mathrm{PF}_n^{\mathrm{inc}}$ be a non-decreasing parking function, and $\ell := \Psi_{\mathrm{PF}\to\mathrm{Luk}}(p)$ the corresponding

Łukasiewicz path. Fix some $j \in \{0, \dots, n-1\}$, and define $\gamma_j := |\{i \in [n];\, p_i \leq j\}|$ to be the number of cars which prefer to park in one of the first j spots. Since p is non-decreasing, Lemma 2.1 and Fact 1.1 then imply that cars 1 to γ_j occupy exactly the spots 1 to $j+h$, where $h := h(\ell; j)$. Now let $k := \ell_{j+1}$, and consider the cars $\gamma_j + 1, \cdots, \gamma_j + k + 1$. By construction, this is exactly the set of cars whose preferred spot is $j+1$. In the parking process for p, these cars occupy spots $j+h+1, \cdots, j+h+k+1$. This yields the (partial) displacement vector

$$\left(d_{\gamma_j+1}, \dots, d_{\gamma_j+k+1}\right) = (h, \dots, h+k). \tag{4}$$

By summation, we get: $d_{\gamma_j+1} + \cdots + d_{\gamma_j+k+1} = h + \cdots + (h+k) = (k+1)h + 1 + \cdots + k = (k+1)h + \frac{k(k+1)}{2} = \frac{(k+1)(2h+k)}{2}$, which is exactly the area $\mathcal{A}(k,h)$ given in Equation (1). In words, the total displacement of cars preferring spot $(j+1)$ is equal to the area under the $(j+1)$-th step of the Łukasiewicz path ℓ, which immediately gives $|\text{disp}\,(p)| = \text{Area}\,(\ell)$ by summing over all steps. Moreover, in Equation (4), we may also take the maximum to get

$$\max\left(d_{\gamma_j+1}, \dots, d_{\gamma_j+k+1}\right) = h + k = h(\ell; j) + \ell_{j+1} = h(\ell; j+1).$$

In other words, for any $j \in \{0, \dots, n-1\}$, the height $h(\ell; j+1)$ of the Łukasiewicz path ℓ after $j+1$ steps is equal to the maximum displacement of cars $\gamma_j + 1, \cdots, \gamma_j + k + 1$, which as noted above are exactly those that prefer spot $j+1$. Taking the maximum over all such j immediately yields $\max\,(\text{disp}\,(p)) = \text{Height}\,(\Psi_{\text{PF}\to\text{Luk}}(p))$, as desired. □

Remark 2.3. *Theorem 2.1 essentially states that non-decreasing parking functions are uniquely defined by the numbers of cars preferring each spot in the car park. In other words, Łukasiewicz paths encode parking functions up to the permutation of their elements. In order to encode all parking functions, we need to also know which cars prefer a given spot. This can be done by labelling each step* $s^k = (k+1, k)$ *for* $k \geq 0$ *of the Łukasiewicz path with a subset* $S \subset [n]$ *of size* $k+1$ *such that the label sets over all steps form a partition of the set* $[n]$*. For example, consider the Łukasiewicz path* ℓ *from Figure 1.3. We may choose the following labelling:* $\left(2^{\{2,3,10\}}, -1, 0^{\{5\}}, 1^{\{6,8\}}, -1, -1, 0^{\{9\}}, 3^{\{1,7,11,12\}}, -1, -1, -1, 0^{\{4\}}\right)$*, with the exponent indicating the label set associated to each step. This encodes the parking function* $p = (8, 1, 1, 12, 3, 4, 8, 4, 7, 1, 8, 8)$*.*

In the above example, we have $\mathcal{O}\,(p) = (8, 1, 2, 12, 3, 4, 9, 5, 7, 6, 10, 11)$, yielding the displacement vector $\text{disp}\,(p) = (0, 0, 1, 0, 0, 0, 1, 1, 0, 5, 2, 3)$, so the maximum displacement is 5, while the path has height 3. This means that in general, the map $\Psi_{\text{PF}\to\text{Luk}}$ does not map the maximum displacement of a parking function to the height of its corresponding path. On the other hand, the total displacement is still equal to 13. We will see that this property holds true in general.

Theorem 2.3. *The map* $\Psi_{\text{PF}\to\text{Luk}} : \text{PF}_n \to \text{Luk}_n$ *is a surjection. Moreover, for any* $p \in \text{PF}_n$*, we have* $|\text{disp}\,(p)| = \text{Area}\,(\Psi_{\text{PF}\to\text{Luk}}(p))$*. Finally, for any Łukasiewicz path* ℓ*, the fibre set* $\Psi^{-1}_{\text{PF}\to\text{Luk}}(\ell) := \{p \in \text{PF}_n;\, \Psi_{\text{PF}\to\text{Luk}}(p) = \ell\}$ *is obtained by taking all possible permutations of the non-decreasing parking function* $\Psi_{\text{Luk}\to\text{PF}}(\ell)$*.*

Proof. The surjectivity of $\Psi_{\text{PF}\to\text{Luk}}$ and description of its fibres follow from Theorem 2.1 and Remark 2.3. We show that for any $p \in \text{PF}_n$, we have $|\text{disp}\,(p)| = \text{Area}\,(\Psi_{\text{PF}\to\text{Luk}}(p))$. In fact, since this formula holds for non-decreasing parking functions by Theorem 2.1, it suffices to show that the total displacement $|\text{disp}\,(p)|$ is invariant under permutation of parking preferences. But this follows essentially from the definition of the displacement, combined with the observation that if $o = \mathcal{O}\,(p) = (o_1, \dots, o_n)$ is the outcome of a parking function p, every spot in $[n]$ appears exactly once in o. Then we get:

$$|\text{disp}\,(p)| = \sum_{i=1}^{n}(o_i - p_i) = \sum_{i=1}^{n} o_i - \sum_{i=1}^{n} p_i = \frac{n(n+1)}{2} - \sum_{i=1}^{n} p_i,$$

and the right-hand side is clearly invariant under permutation. □

3. Specialisations

There are a number of natural restrictions that we can place on parking functions. For example, if we restrict each car to have displacement at most one, we get so-called *unit-interval* parking functions (see Section 3.3 for a short discussion on what is known about these). Conversely, Łukasiewicz paths can also be restricted, for example in terms of their height, or the largest step size allowed. In this section, we study several such restrictions under the bijections $\Psi_{\text{PF}\to\text{Luk}}$ and $\Psi_{\text{Luk}\to\text{PF}}$.

3.1. The Motzkin family

If we restrict steps in a Łukasiewicz path to have size at most 1, we get the well-known *Motzkin paths* (see e.g. [2]). We denote by Motz_n the set of Motzkin paths with n steps. These are counted by the Motzkin numbers (Sequence A001006 in the OEIS [19]). The corresponding parking functions $p = (p_1, \dots, p_n) \in [n]^n$ are those that satisfy the restriction

$$\forall i \in [n],\ |\{j \in [n];\, p_j = i\}| \leq 2. \tag{5}$$

In other words, every spot in the car park is preferred by at most 2 cars. These were studied in previous work by the authors [15, Section 3] under the name of *Motzkin parking functions*. In particular, they provided a bijection between non-crossing matchings and parking functions whose *MVP outcome* reverses the order of the cars. Here, the MVP outcome of a parking function is the order in which cars end up parking if they follow the MVP (Most Valuable Player) parking process defined by Harris et al. [11]. In this process, when a car finds its preferred spot occupied by a previous car, it "bumps" that car out of the spot and parks there. The earlier car then has to drive on, and park in the first available spot it can find.

3.2. Prime parking functions

Given a parking function $p = (p_1, \ldots, p_n) \in \mathrm{PF}_n$, and an index $j \in [n]$, we say that j is a *breakpoint* for p if $|\{i \in [n];\, p_i \leq j\}| = j$, i.e. exactly j cars prefer the first j spots. A parking function is said to be *prime* if its only breakpoint is at index n. The concept of prime parking functions was introduced by Gessel, who showed that there are $(n-1)^{(n-1)}$ prime parking functions (see e.g. [17, Exercise 5.49]). A bijective proof of this formula was later given in [7]. We denote by $\mathrm{PrimePF}_n$, respectively $\mathrm{PrimePF}_n^{\mathrm{inc}}$, the set of prime parking functions, respectively non-decreasing prime parking functions, of length n. In general, breakpoints of parking functions are easily read from the corresponding Łukasiewicz path.

Proposition 3.1. *Let $p \in \mathrm{PF}_n$ be a parking function and $j \in [n]$ an index. Then j is a breakpoint for p if and only if the Łukasiewicz path $\ell := \Psi_{\mathrm{PF}\to\mathrm{Luk}}(p)$ hits the x-axis after j steps, i.e. $h(\ell; j) = 0$.*

Proof. By construction, if $\ell = (\ell_1, \ldots, \ell_n)$, then the height of the path after j steps is simply

$$h(\ell; j) = \sum_{i=1}^{j} \ell_i = \sum_{i=1}^{j} (|\{k \in [n];\, p_k = i\}| - 1) = |\{i \in [n];\, p_i \leq j\}| - j,$$

where we applied the definition of $\Psi_{\mathrm{PF}\to\mathrm{Luk}}$ from Equation (2). The result immediately follows. □

We say that a Łukasiewicz path $\ell \in \mathrm{Luk}_n$ is *prime* if it stays strictly above the x-axis other than at its start and end points $(0,0)$ and $(n,0)$, and denote by $\mathrm{PrimeLuk}_n$ the set of prime Łukasiewicz paths. We now state our first specialisation.

Theorem 3.1. *The map $\Psi_{\mathrm{PF}\to\mathrm{Luk}} : \mathrm{PrimePF}_n^{\mathrm{inc}} \to \mathrm{PrimeLuk}_n$ is a bijection. Also, $|\mathrm{PrimePF}_n^{\mathrm{inc}}| = |\mathrm{PrimeLuk}_n| = \mathrm{Cat}_{n-1}$.*

Proof. That $\Psi_{\mathrm{PF}\to\mathrm{Luk}}$ induces a bijection from non-decreasing prime parking functions to prime Łukasiewicz paths is an immediate consequence of Proposition 3.1. To get the enumeration, notice that a non-decreasing parking sequence $p = (p_1, \ldots, p_n)$ is a prime parking function if and only if we have $p_1 = 1$ and $p_i < i$ for all $i \geq 2$ (applying Proposition 1.1, Case (2), and the definition of prime parking functions). This immediately implies that the map $(p_1, \ldots, p_n) \mapsto (p_2, \ldots, p_n)$ is a bijection from $\mathrm{PrimePF}_n^{\mathrm{inc}}$ to $\mathrm{PF}_{n-1}^{\mathrm{inc}}$, yielding the desired enumeration. □

Remark 3.1. *We can also express the above bijection $\mathrm{PrimePF}_n^{\mathrm{inc}} \to \mathrm{PF}_{n-1}^{\mathrm{inc}}$ in terms of Łukasiewicz paths. We get the bijection $\mathrm{PrimeLuk}_n \to \mathrm{Luk}_{n-1}$, $(\ell_1, \ldots, \ell_n) \mapsto (\ell_1 - 1, \ell_2, \ldots, \ell_{n-1})$, from prime Łukasiewicz paths of length n to Łukasiewicz paths of length $n-1$. In words, given a prime Łukasiewicz path, we decrease the size of its first step by one, and delete the last step (this is necessarily a "down" step, since the path is prime), yielding a Łukasiewicz path with one less step. This bijection is illustrated in Figure 3.1 for the prime Łukasiewicz path $\ell = (3, -1, -1, 1, 0, -1, 0, -1)$ (left), which maps to the Łukasiewicz path $\ell' = (2, -1, -1, 1, 0, -1, 0)$ (right).*

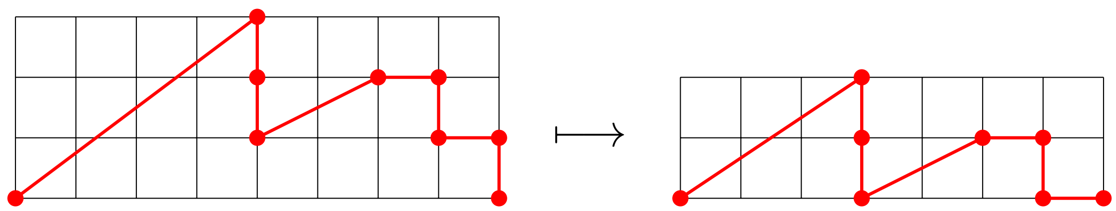

Figure 3.1: Illustrating the bijection from $\mathrm{PrimeLuk}_n$ to Luk_{n-1}.

3.3. Unit-interval parking functions

Definition 3.1. *A parking function p is said to be* unit-interval *if it satisfies* $\max(\mathrm{disp}(p)) \leq 1$.

In other words, a unit-interval parking function is one where each car has displacement zero or one. Unit-interval parking functions were originally defined by Hadaway [10], who showed that they are enumerated by the Fubini numbers.

The theory of unit-interval parking functions was further developed in [1,3,8,14], with rich combinatorial connections to the permutohedron and a generalisation of the aforementioned Fubini numbers. We denote by UPF_n, respectively $\mathrm{UPF}_n^{\mathrm{inc}}$, the set of unit-interval parking functions, respectively unit-interval non-decreasing parking functions, of length n. We first state a straightforward characterisation of unit-interval parking functions.

Proposition 3.2. *Let $p = (p_1, \ldots, p_n) \in \mathrm{PF}_n$ be a parking function with outcome $\mathcal{O}(p) = (o_1, \ldots, o_n)$. Then $p \in \mathrm{UPF}_n$ if and only if we have $o_i \in \{p_i, p_i + 1\}$ for all $i \in [n]$. That is, each car parks either in its preferred spot, or in the spot immediately after.*

Now consider a unit-interval parking function $p \in \mathrm{UPF}_n$, and its corresponding Łukasiewicz path $\ell := \Psi_{\mathrm{PF}\to\mathrm{Luk}}(p)$. By Theorem 2.1 and Definition 3.1, we have $\mathrm{Height}(\ell) = \max(\mathrm{disp}(p)) \leq 1$. In particular, all steps in ℓ have size at most one, making ℓ a Motzkin path, or equivalently p is a Motzkin parking function in the sense of Section 3.1. This can also be seen directly from the parking function p. Indeed, in any parking function, if three cars prefer the same spot, then the last of these cars to arrive must have displacement at least two. We then get the following specialisation. To simplify notation, we write $\mathrm{Motz}_n^{\leq 1}$ for the set of Motzkin paths of length n and height at most one, and refer to these as 1-Motzkin paths.

Theorem 3.2. *The map $\Psi_{\mathrm{PF}\to\mathrm{Luk}} : \mathrm{UPF}_n^{\mathrm{inc}} \to \mathrm{Motz}_n^{\leq 1}$ is a bijection. Moreover, we have $|\mathrm{UPF}_n^{\mathrm{inc}}| = |\mathrm{Motz}_n^{\leq 1}| = 2^{n-1}$.*

Proof. That the map is a bijection follows from the preceding remarks and Theorem 2.1. For the enumeration, note that if $p \in \mathrm{PF}_n^{\mathrm{inc}}$, then p is unit-interval if and only if we have $p_1 = 1$, and $p_i \in \{i-1, i\}$ for all $i \geq 2$ (applying Proposition 3.2). There are therefore two choices for each car 2 to n, yielding 2^{n-1} choices in total. □

The enumeration can also be seen on the Motzkin paths. Indeed, a 1-Motzkin path m is uniquely characterised by the subset $\{j \in [n-1];\, h(m;j) = 1\}$, giving a bijection between $\mathrm{Motz}_n^{\leq 1}$ and subsets of $[n-1]$.

Remark 3.2. *Unlike prime parking functions, the set of unit-interval parking functions is not* permutation invariant*. That is, there exists a unit-interval parking function p such that permuting the preferences of p no longer yields a unit-interval parking function. In the notation of this paper, this means that there exists a parking function $p \in \mathrm{PF}_n \setminus \mathrm{UPF}_n$ such that $\Psi_{\mathrm{PF}\to\mathrm{Luk}}(p)$ is a 1-Motzkin path. For example, if we take $p = (1, 2, 1) \in \mathrm{PF}_3$, then car 3 has displacement 2, so p is not unit-interval, but $\Psi_{\mathrm{PF}\to\mathrm{Luk}}(p)$ is the Łukasiewicz path with steps $(1, 0, -1)$, which is 1-Motzkin. In particular, this means that there are labellings of a 1-Motzkin path, in the sense of Remark 2.3, which do not yield unit-interval parking functions.*

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