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*Research Article*

# Zykov sums of digraphs with diachromatic number equal to its harmonious number

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**Abstract**

The dichromatic number and the diachromatic number are generalizations of the chromatic number and the achromatic number for digraphs considering acyclic colorings. In this paper, we determine the diachromatic number of digraphs arising from the Zykov sum of digraphs that accept a complete $k$-coloring with $k = \frac{1+\sqrt{1+4m}}{2}$ for a suitable $m$. As a consequence, the diachromatic number equals the harmonious number for every digraph in this family. In particular, we determine the diachromatic number of digraphs arising from the Zykov sum of Hamiltonian factorizations of complete digraphs over a suitable digraph. We also obtain the equivalent results for graphs. Furthermore, we determine the achromatic number for digraphs arising from the generalized composition in terms of the thickness of complete graphs. Finally, we extend some results on the dichromatic number of Zykov sums of tournaments to the class of digraphs that are not tournaments and apply them, and the results obtained for the diachromatic number, to the problem of the existence of a digraph with dichromatic number $r$ and diachromatic number $t$ for some particular cases with $2 \leq r \leq t$.

**Keywords:** diachromatic number; dichromatic number; achromatic number; harmonious number; factorization; products of (di)graphs.

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## 1. Introduction

A *$k$-coloring* of a digraph $D$ is an acyclic vertex-coloring (that is, each color class induces a subdigraph with no directed cycles). The *dichromatic number* $dc(D)$ of $D$ is the smallest $k$ for which there exists a $k$-coloring of $D$ [10]. Any $dc(D)$-coloring of $D$ is also *complete* (that is, for every pair $(i, j)$ of different colors there is at least one arc $uv$ such that $u$ is colored $i$ and $v$ is colored $j$) [2,5–7]. The *diachromatic number* $dac(D)$ of $D$ is the largest $k$ in a complete $k$-coloring of $D$ [1]. Therefore, the size $m$ of a digraph $D$ is upper bounded by $2\binom{dac(D)}{2}$ and hence,

$$dac(D) \leq \tfrac{1+\sqrt{1+4m}}{2} \tag{1}$$

and both coincide if and only if there are exactly two arcs between both color classes. For graphs (which can be seen as symmetric digraphs), such parameters are called the *chromatic number* $\chi$ and the *achromatic number* $\psi$, respectively.

On the other hand, a coloring of $D$ is called *harmonious* if for every pair $(i, j)$ of different colors there is at most one arc $uv$ such that $u$ is colored $i$ and $v$ is colored $j$ [5–7]. The *harmonious number* $dh(D)$ of $D$ is the smallest $k$ for which there exists a harmonious $k$-coloring of $D$. For graphs, it is called *harmonious number* and it is denoted by $h$. Furthermore, observe that the size $m$ of a digraph $D$ is bounded above by $2\binom{dh(D)}{2}$. Therefore, for any digraph $D$ of size $m$ we have that

$$dc(D) \leq dac(D) \leq \tfrac{1+\sqrt{1+4m}}{2} \leq dh(D). \tag{2}$$

Observe that the first inequality becomes equality whenever the minimum and maximum numbers of colors in an acyclic complete coloring coincide, as in the case of the directed triangle; the second and third inequalities become equalities when there is exactly one $(i, j)$-arc for each pair of colors $i \neq j$.

Let $D$ be a digraph and $X = \{H_u : u \in V(D)\}$ a family of nonempty mutually vertex-disjoint digraphs. The *Zykov sum* $\sigma(X, D)$ of $X$ over $D$ is a digraph with vertex set $\bigcup_{u\in V(D)} V(H_u)$ and arc set

$$\bigcup_{u\in V(D)} A(H_u) \cup \{ab : a \in V(H_u), b \in V(H_v), uv \in A(D)\}.$$

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The corresponding operation for graphs is called *generalized composition* and it is denoted by $D[X]$. If $H_u \cong H$ for every $u \in V(D)$, then $\sigma(X, D)$ is called *lexicographic product* (also called *digraph composition*) and it is denoted by $D[H]$. Also, $D^i[H] := (D^{i-1}[H])[H]$ with $D^1[H] := D[H]$. The dichromatic number of Zykov sums and composition of digraphs were studied by Neumann-Lara in [11].

In this paper, we determine the diachromatic number of digraphs arising from the Zykov sum of digraphs that accept a complete $k$-coloring with $k = \frac{1+\sqrt{1+4m}}{2}$ for a suitable $m$. As a consequence, the diachromatic number equals the harmonious number for every digraph in this family. We also determine the diachromatic number (and hence the harmonious number) of digraphs arising from the Zykov sum of Hamiltonian factorizations of complete digraphs over a suitable digraph. Finally, we extend some results on the dichromatic number of Zykov sums of tournaments to the class of digraphs (which are not tournaments) and apply the results obtained in Section 3 to the problem of the existence of digraphs with dichromatic number $r$ and diachromatic number $t$ for some particular cases with $2 \leq r \leq t$.

## 2. Definitions

For concepts not defined here, we refer the reader to [4]. Let $[n]$ denote the set $\{1, 2, \ldots, n\}$. For two nonempty vertex sets $X, Y$ of a digraph $D$, we define $[X, Y] = \{(x, y) \in A(D) \mid x \in X, y \in Y\}$. Let $m \geq 2$. In the case of digraphs, $K_m$ denotes the complete symmetric digraph. In the case of graphs, $K_m$ denotes the complete graph. A *factor* $H_j$ of the complete digraph (respectively, graph) $K_m$ is a spanning subdigraph (respectively, subgraph). A *factorization* $Y$ of $K_m$ is a set of $q$ pairwise arc-disjoint (respectively, edge-disjoint) factors $H_j$ such that these factors induce a partition in the arcs (respectively, edges) with $j \in [q]$. If $H_j \cong H$ (for all $j \in [q]$) then it is called $H$-*factorization*. A *relabel factorization* $X$ of a factorization $Y$ is to relabel the vertices $v^1, v^2, \ldots, v^m$ of each factor $H_j$ into $v_j^1, v_j^2, \ldots, v_j^m$ to make pairwise-disjoint vertices. Let $D$ be a $k$-diachromatic digraph (respectively, $k$-achromatic graph) with a $k$ coloring $\varphi$ and, let $\{C_1, C_2, \ldots, C_k\}$ be the set of chromatic classes for $\varphi$ with $|C_i| = q_i$. For each $i \in [k]$, denote the set of vertices of the chromatic class $C_i$ by $\{u_{i,1}, u_{i,2}, \ldots u_{i,q_i}\}$. In this case, $V(D) = \bigcup_{i=1}^{k} C_i$. For each $i \in [k]$, let $X_i = \{H_{u_{i,1}}, H_{u_{i,2}}, \ldots, H_{u_{i,q_i}}\}$ be a relabel factorization of $K_{m_i}$ into $q_i$ factors; that is, $\bigcup_{j=1}^{q_i} H_{u_{i,j}} = K_{m_i}$. We consider the Zykov sum $\sigma(X, D)$, where

$$X = (H_{u_{i,j}})_{u_{i,j} \in V(D)} = \bigcup_{i=1}^{k} X_i.$$

Observe that for the vertex $v_{i,j}^l$, the subindex $j$ and the superindex $l$ correspond to vertex $l$ in the factor $j$ of the relabel factorization of the complete digraph (respectively, graph) $K_{m_i}$ and the subindex $i$ corresponds to the color of the vertex $u_{i,j}$ in the digraph (respectively, graph) $D$. For the ease of reading, in Figure 2.1 we depict the Zykov sum $\sigma(X, \overrightarrow{C}_6)$, where $X = \{X_1, X_2, X_3\}$ while $X_1$, $X_2$ and $X_3$ are relabel factorizations of $K_2$, $K_3$ and $K_4$ respectively, the color 1, 2 and 3 are represented in the vertices by the symbols ★, ♦ and ▲, respectively.

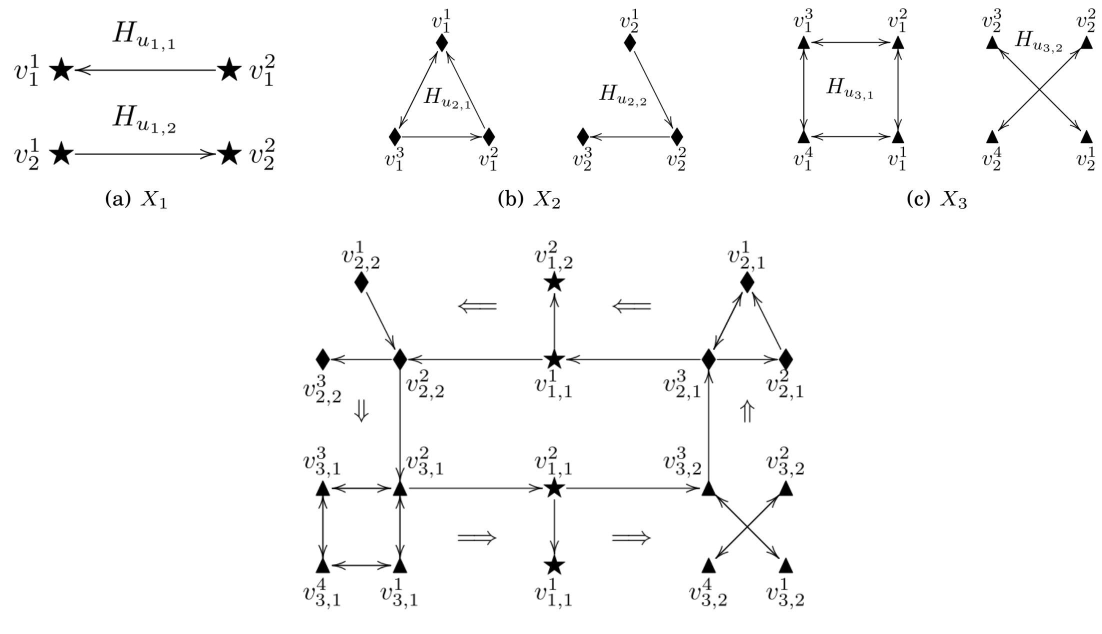

Figure 2.1: The Zykov sums.

An *equitable coloring* is a coloring in such a way that the numbers of vertices in any two color classes differ by at most one [9]. We reserve the term *balanced coloring* for the case in which any two color classes have the same cardinality.

## 3. The diachromatic number of Zykov sums

Let $m \geq 2$. Throughout this section, $K_m$ denotes the complete symmetric digraph. A digraph $D$ is $k$-*minimal* if $dac(D) = k$ and $dac(D - f) < k$ for all $f \in A(D)$.

**Theorem 3.1** (see [1]). *Let $D$ be a digraph with diachromatic number $k$. Then, $D$ is $k$-minimal if and only if $D$ has size $k(k-1)$.*

**Theorem 3.2.** *Let $D$ be a $k$-minimal digraph of order $n$ with a $k$-coloring $\varphi$. Let $\{C_1, C_2, \ldots, C_k\}$ be the set of chromatic classes for $\varphi$ with $|C_i| = q_i$. For each $i \in [k]$, let $C_i = \{u_{i,1}, u_{i,2}, \ldots u_{i,q_i}\}$ and let $X_i = \{H_{u_{i,1}}, H_{u_{i,2}}, \ldots, H_{u_{i,q_i}}\}$ be a relabel factorization of $K_{m_i}$ into $q_i$ factors. Then $\sigma(X, D)$ is $t$-minimal, where*

$$X = \bigcup_{i=1}^{k} X_i \quad \textit{and} \quad t = \sum_{i=1}^{k} m_i.$$

**Proof.** We take a partition of $K_{m_i}$ in $q_i$ factors. In order to have a set of colored and sorted vertices arising from $V(K_{m_i}) = \{v_i^1, v_i^2, \ldots, v_i^{m_i}\}$, we define the following coloring. Let $f_i \colon V(K_{m_i}) \to [m_i]$ be the complete $m_i$-colorings of $K_{m_i}$ such that $f_i(v_i^l) = l$ for each $l \in [m_i]$. Let $f_{i,j} \colon V(H_{u_{i,j}}) \to [m_i]$ the natural restriction of $f_i$ into each factor $H_{u_{i,j}}$, that is, $f_{i,j}(v_{i,j}^l) = f_i(v_i^l) = l$ for any vertex $v_{i,j}^l \in V(H_{u_{i,j}})$, with $i \in [k]$, $j \in [q_i]$ and $l \in [m_i]$, see Figure 2.1.

Let $\varsigma \colon V(\sigma(X, D)) \to [t]$ be a $t$-coloring such that for each $l \in [m_i]$

$$\varsigma(v_{i,j}^l) = c(i,l) := \sum_{a=0}^{i-1} m_a + l, \text{with } m_0 = 0.$$

That is, if $i$ and $l$ are fixed, for each $j \in [q_i]$ the vertex $v_{i,j}^l$ in the factor $H_{u_{i,j}}$ has color $c(i,l)$. Thus, the set of vertices colored $c(i,l)$ of $\varsigma$ is

$$\{v_{i,1}^l, v_{i,2}^l, \ldots, v_{i,q_i}^l\}.$$

Since the Zykov sums of empty graphs is empty, the coloring is proper and then acyclic due to the fact that the induced subgraph by $\{v_{i,1}^l, v_{i,2}^l, \ldots, v_{i,q_i}^l\}$ of $\sigma(X, D)$ is empty.

Next, we claim the $\varsigma$ coloring is minimal and complete. Let $c(i,l)$ and $c(i',l')$ be two colors of $\varsigma$ with $i, i' \in [q_i]$, $l \in [m_i]$ and $l' \in [m_{i'}]$. If $i = i'$, since each $H_{u_{i,j}}$ has the $m_i$ colors of $f_i$, then $v_{i,j}^l v_{i,j}^{l'}$ is the unique arc of $H_{u_{i,j}}$ for some $j$ and then there exists a unique arc between $c(i,l)$ and $c(i,l')$. On the other hand, since $\varphi$ is minimal and complete, if $i \neq i'$ there exists a unique arc $u_{i,j}u_{i',j'}$ such that $\varphi(u_{i,j}) = i$ and $\varphi(u_{i',j'}) = i'$ with $j \in [q_i]$ and $j' \in [q_i']$. Therefore, $\left[V(H_{u_i,j}), V(H_{u_i',j'})\right]$ is a bipartition of a directed complete bipartite subdigraph of $\sigma(X, D)$. In consequence, for a fixed $l$ and $l'$ the arc $v_{i,j}^l v_{i',j'}^{l'}$ is the unique arc from a vertex of color $c(i,l)$ to a vertex with color $c(i',l')$. □

The following corollaries are direct consequences of Theorem 3.2.

**Corollary 3.1.** *Let $D$ be a $k$-minimal digraph of order $n$ with a equitable $k$-coloring $\varphi$. Let $X_i$ be a relabel factorization of $K_m$ into $q$ factors, that is, $X_i = \{H_{u_{i,1}}, H_{u_{i,2}}, \ldots, H_{u_{i,q}}\}$ for $i \in [k]$. Then $\sigma(X, D)$ is $km$-minimal with an equitable $km$-coloring where*

$$X = \bigcup_{i=1}^{k} X_i.$$

**Corollary 3.2.** *Let $D$ be a $k$-minimal digraph of order $n$ with a balanced $k$-coloring, such that $qk = n$. If $K_m$ has a relabel $H$-factorization into $q$ factors, then $D[H]$ is $km$-minimal with a balanced $km$-coloring.*

Note that Theorem 3.2 produces a $t$-minimal digraph for which, their chromatic classes $\{v_{i,1}^l, v_{i,2}^l, \ldots, v_{i,q_i}^l\}$ have cardinality equal to $U_i$, therefore this digraph and the $X_i$ relabel factorization ($m_i$ copies) fulfills the hypothesis, hence, a recursive construction can be done given an initial digraph $D$ and $X_i$ factorizations.

**Corollary 3.3.** *Let $D$ be a $k$-minimal digraph of order $n$ with a balanced $k$-coloring, such that $qk = n$. If $K_m$ has a relabel $H$-factorization into $q$ factors, then $D^i[H]$ is $k^i m$-minimal with a balanced $k^i m$-coloring for all $i \in \mathbb{Z}^+$.*

Now, we proceed to construct families of digraphs obtained by Zykov sums $D$ and $H$ that satisfy the hypothesis of Theorem 3.2. We recall some definitions given in [1]. Two vertices are *adjacent* if they are in a 2-cycle. To obtain an *elementary dihomomorphism* of a digraph $D$, identify two nonadjacent vertices $u$ and $v$ of $D$. The resulting vertex when identifying $u$ and $v$ could be denoted by either $u$ or $v$. An elementary dihomomorphism preserving the cardinality of arcs is called *elementary identification* $\epsilon$, that is, let $D$ be a digraph and $u, v \in V(D)$ two independent vertices such that $N^+(u) \cap N^+(v) = \emptyset$ and $N^-(u) \cap N^-(v) = \emptyset$, then $\epsilon$ is the elementary dihomomorphism obtained by identifying $u$ and $v$. A digraph $D'$ is an identification image of a digraph $D$ if and only if $D'$ can be obtained by a sequence of elementary identifications beginning with $D$.

An *elementary unfold* is the inverse image of an elementary identification and an *unfold* is the inverse image of an identification. For example, an unfold of $K_5$ is $\overrightarrow{C}_{20}$ if we follow a Eulerian circuit of $K_5$, and vice versa, an identification of $\overrightarrow{C}_{20}$ is $K_5$.

**Remark 3.1.** *A digraph $D$ is $k$-minimal if and only if there exists an elementary identification $\Gamma$ from the digraph $D$ to the complete digraph $K_k$.*

As a direct consequence, we have the following theorem.

**Theorem 3.3.** *Let $\overrightarrow{C}_n$ with $n \geq 0$. The $dac(\overrightarrow{C}_n) = k$ if $k(k-1) \leq n < k(k+1)$. Moreover $\overrightarrow{C}_n$ is $k$-minimal if and only if $n = k(k-1)$.*

As a consequence of Remark 3.1 and Theorem 3.3, we have that $K_k$ can be unfolded in the cycle $\overrightarrow{C}_{k(k-1)}$. Consider an unfold in $K_{2q+1}$ into $\overrightarrow{C}_{2q(2q+1)}$. The induced $k$-coloring of $\overrightarrow{C}_{4q^2+2q}$, for $k = 2q+1$, is equitable where each chromatic class has $2q$ vertices. On the other hand, for $m = 2q+1$, it is known that $K_m$ accepts an $H$-factorization into $q$ factors where $H$ is a Hamiltonian cycle.

**Corollary 3.4.** *The digraph $D = \overrightarrow{C}^{\,i}_{4q^2+2q}[\overrightarrow{C}_{2q+1}]$ is $(2q+1)^{i+1}$-minimal with a balanced $(2q+1)^{i+1}$-coloring, then for all $i, q \in \mathbb{Z}^+$*

$$dac(D) = dh(D) = (2q+1)^{i+1}.$$

Now, for $m = 2q$, it is known that $K_m$ accepts an $H$-factorization into $q$ factors, where $H$ is a Hamiltonian path. Hence, we have the following corollary:

**Corollary 3.5.** *The digraph $D = \overrightarrow{C}^{\,i}_{4q^2+2q}[\overrightarrow{P}_{2q}]$ is $2q(2q+1)^i$-minimal with a balanced $2q(2q+1)^i$-coloring, then for all $i, q \in \mathbb{Z}^+$*

$$dac(D) = dh(D) = 2q(2q+1)^i.$$

## 4. The achromatic number of generalized compositions of graphs

In this section, we extend the results of Section 3 for graphs. Since the proofs of these results for graphs are analogous to those of the results for digraphs, we omit them.

**Theorem 4.1.** *Let $G$ be a digraph with achromatic number $k$. Then, $G$ is $k$-minimal if and only if $G$ has size $\binom{k}{2}$.*

**Theorem 4.2.** *Let $G$ be a $k$-minimal graph of order $n$ with a $k$-coloring $\varphi$. Let $\{C_1, C_2, \dots, C_k\}$ be the set of chromatic classes for $\varphi$ with $|C_i| = q_i$. For each $i \in [k]$ let $C_i = \{u_{i,1}, u_{i,2}, \dots u_{i,q_i}\}$ and let $X_i = \{H_{u_{i,1}}, H_{u_{i,2}}, \dots, H_{u_{i,q_i}}\}$ be a relabel factorization of $K_{m_i}$ into $q_i$ factors. Then $G[X_i]$ is $t$-minimal, where*

$$X = \bigcup_{i=1}^{k} X_i \quad \textit{and} \quad t = \sum_{i=1}^{k} m_i.$$

**Corollary 4.1.** *Let $G$ be a $k$-minimal graph of order $n$ with a balanced $k$-coloring $\varphi$. Let $X_i$ be a relabel factorization of $K_m$ into $q_i$ factors. Then $G[X_i]$ is $km$-minimal with an balanced $km$-coloring, where*

$$X = \bigcup_{i=1}^{k} X_i.$$

**Corollary 4.2.** *Let $G$ be a $k$-minimal graph of order $n$ with a balanced $k$-coloring, such that $qk = n$ for some $q \in \mathbb{N}$. If $K_m$ has a relabel $H$-factorization into $q$ factors, then $G[H]$ is $km$-minimal with a balanced $km$-coloring.*

**Corollary 4.3.** *Let $G$ be a $k$-minimal graph of order $n$ with a balanced $k$-coloring, such that $qk = n$ for some $q \in \mathbb{N}$. If $K_m$ has a relabel $H$-factorization into $q$ factors, then $G^i[H]$ is $k^i m$-minimal with a balanced $k^i m$-coloring for all $i \in \mathbb{Z}^+$.*

For the case of graphs, take an unfold of $K_{2q+1}$, following an Eulerian circuit, into $C_{q(2q+1)}$. The induced $k$-coloring of $C_{2q^2+q}$, for $k = 2q+1$, is equitable where each class has $q$ vertices then we have the following corollaries. Note that $C_{4q^2+2q}$ is an unfold of $C_{2q^2+q}$ and two empty disjoint directed cycles of the same size can be identified in a cycle.

**Corollary 4.4.** *The graph $G = C^i_{2q^2+q}[C_{2q+1}]$ is $(2q+1)^{i+1}$-minimal with a balanced $(2q+1)^{i+1}$-coloring, then for all $i, q \in \mathbb{Z}^+$*

$$\psi(G) = h(G) = (2q+1)^{i+1}.$$

**Corollary 4.5.** *The graph $G = C^i_{2q^2+q}[P_{2q}]$ is $2q(2q+1)^i$-minimal with a balanced $2q(2q+1)^i$-coloring, then for all $i, q \in \mathbb{Z}^+$*

$$\psi(G) = h(G) = 2q(2q+1)^i.$$

## 5. Applications

Consider the following result by V. Bhave:

**Theorem 5.1** (see [3])**.** *For every pair of integers $a \leq b$, there exists a graph $G$ such that $\chi(G) = a$ and $\alpha(G) = b$.*

Observe that for symmetric digraphs this result can be extended trivially since the bidirected orientation $\overleftrightarrow{G}$ of any graph $G$ satisfies that $dc(\overleftrightarrow{G}) = \chi(G)$ and $dac(\overleftrightarrow{G}) = \alpha(G)$.

In order to extend Theorem 5.1 to the class of non-symmetric digraphs, we use results of Section 3 to establish some sets of integers $a, b$ such that there exists a digraph $D$ with $dc(D) = a$ and $dac(D) = b$. In Corollary 3.4, for any two positive integers $i, q$, we determined the diachromatic number of the composition $D = \overrightarrow{C}_{4q^2+2q}[\overrightarrow{C}^i_{2q+1}]$. In order to determine the dichromatic number of $D = \overrightarrow{C}_{4q^2+2q}[\overrightarrow{C}^i_{2q+1}]$, we follow the ideas proposed by Neumann-Lara in [11]. Since Neumann-Lara studied tournaments and a tournament is acyclic if and only if it is transitive, these two concepts are equivalent in the class of tournaments, thus for tournaments, if a chromatic class is transitive it is clearly acyclic, but for digraphs, we only require that the chromatic classes are acyclic.

The following result is a generalization of Propositions 32$(iii)$ and 34 [11], simplifying the notation using Corollary 43 [11]. We omit the proof because it is analogous to the original one (changing transitive sets by acyclic sets, tournaments by digraphs, and $\Lambda_{m,r}$ by $\Lambda'_{m,r}$).

**Proposition 5.1.** *Let $H, \alpha$ be digraphs such that $H$ has order $m$ and $dc(\alpha) = k$, then*

1. $dc(H[\alpha]) \geq \lceil \frac{k \cdot m}{r} \rceil$.

*Let $r$ be the maximum order of an acyclic set of vertices of $H$. If $H$ contains an isomorphic copy of $\Lambda'_{m,r}$ as a spanning subgraph, then*

2. $dc(H[\alpha]) = \lceil \frac{k \cdot m}{r} \rceil$.

The next result is concerned with the recurrence relation that appears in the solution of the legendary Josephus Flavius problem. For more details about the mathematical problem see [12]. The approach is similar to the one used in [8].

**Theorem 5.2** (see Theorem 1 in [12])**.** *Consider the recurrence relation $D^{2q+1}_n = \left\lceil \frac{2q+1}{2q} \cdot D^{2q+1}_{n-1} \right\rceil$ ($n \geq 1$ and $D^{2q+1}_0 = 1$). For each interger $q \geq 2$ there is real number $K_{2q+1}$ such that*

$$D^{2q+1}_i = K_{2q+1} \left( \frac{2q+1}{2q} \right)^i + e_{i,2q+1}$$

*and $-2q+1 < e_{i,2q+1} \leq 0$.*

The problem of determining the "exact" formula is still open but for $q = 1$

**Corollary 5.1** (see Corollary 1 in [12])**.** *Consider the recurrence relation $D_n = \left\lceil \frac{3}{2} D_{n-1} \right\rceil$ ($n \geq 1$ and $D_0 = 1$), then*

$$D_n = K \left( \frac{3}{2} \right)^n \qquad (n = 1, 2, \dots)$$

*where $K \sim 1.62227$ is an irrational number.*

We define $\overrightarrow{C}^{i}_{2q+1} = \underbrace{\left[\overrightarrow{C}_{2q+1}\left[\overrightarrow{C}_{2q+1}\left[\dots\left[\overrightarrow{C}_{2q+1}\right]\right]\right]\right]}_{i}$.

**Proposition 5.2.** $dc(\overrightarrow{C}^{i}_{2q+1}) = D^{2q+1}_{i}$

**Proof.** Clearly the maximal set of an acyclic set of vertices of $\overrightarrow{C}^{i}_{2q+1}$ is $2q$ and $\overrightarrow{C}^{i}_{2q+1}$ contains an isomorphic copy of $\Lambda'_{2q+1,2q}$ as a spanning subdigraph. By Proposition 5.1, it follows that $dc\left(\overrightarrow{C}_{2q+1}\left[\overrightarrow{C}_{2q+1}\right]\right) = \left\lceil\frac{2(2q+1)}{2q}\right\rceil$ and since $\left\lceil\frac{2q+1}{2q}\right\rceil = 2$, thus

$$dc\left(\overrightarrow{C}_{2q+1}\left[\overrightarrow{C}_{2q+1}\right]\right) = \left\lceil\frac{(2q+1)}{2q}\left\lceil\frac{(2q+1)}{2q}\right\rceil\right\rceil.$$

Repeating this argument $i-1$ times, it follows that

$$dc(\overrightarrow{C}^{i}_{2q+1}) = \underbrace{\left\lceil\frac{2q+1}{q}\left\lceil\frac{2q+1}{q}\left\lceil\dots\left\lceil\frac{2q+1}{q}\right\rceil\right\rceil\right\rceil\right\rceil}_{i}.$$

Now, the result follows from Theorem 5.2. □

From Propositions 5.1 and 5.2, the next result follows.

**Theorem 5.3.** $dc(\overrightarrow{C}_{4q^2+2q}[\overrightarrow{C}^{i}_{2q+1}]) = \left\lceil\frac{4q^2+2q}{4q^2+2q-1}\left\lceil D^{2q+1}_{i}\right\rceil\right\rceil$.

Observe that $\overrightarrow{C}_{4q^2+2q}[\overrightarrow{C}^{i}_{2q+1}]$ is isomorphic to $\overrightarrow{C}^{i}_{4q^2+2q}[\overrightarrow{C}_{2q+1}]$. Therefore, for any pair of positive integers $i$ and $q$, Theorem 5.3 determines the dichromatic number and Corollary 3.4 determines the diachromatic number of the composition $\overrightarrow{C}_{4q^2+2q}[\overrightarrow{C}^{i}_{2q+1}]$. Although these results provide an infinite number of pairs of integers $a \le b$ such that there exists a non-symmetric digraph $D$ satisfying that $dc(D) = a$ and $dac(G) = b$, we do not know the "exact" formula of the dichromatic number; so, we need other methods in order to extend Theorem 5.1 to the class of non-symmetric digraphs.

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