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Research Article

Further remarks on $(0,1)$ Toeplitz determinants

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Abstract

We discuss the computation of the determinants of the $n \times n$ matrices which have the 1 at every (i,j) entry such that $j-i \in \{-2,-1,0,1,t\}$ and 0 everywhere else, in several special cases depending on t and n.

Keywords: Toeplitz matrices; determinants; linear recurrences.

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1. Introduction

This paper is concerned with the quantity $\Delta(n,t)$, which is defined, for all integers $t \geqslant 2$ and $n > t$, as the determinant of the $n \times n$ matrix whose (i,j) entry equals 1 if $j-i \in \{-2,-1,0,1,t\}$ and 0 otherwise. Particularly, this paper makes an addition to the line of the study initiated with several conjectures in [2] regarding potential explicit formulas for the sequences $\Delta(n,n-1)$ and $\Delta(n,n-2)$. These conjectures turned out to be very simple, and the author did not manage to proceed with the official publication of his earlier article containing their proofs (see [1, 6–8] for the approaches of other researchers). Indeed, the present author was advised to make the initial paper 'more substantial', and he went on to extend the conjectured result to the sequence $\Delta(n,n-c)$ with any fixed integer c in [9]. The computer calculations performed during the work on [9] revealed several unexpected regularity patterns in the values of $\Delta(n,t)$, so the author was tempted to formulate several further conjectures on their behavior. In particular, it was observed that $\Delta(8k+2,4k+2) = k^2$ and $\Delta(8k,4k) = k^2+1$, and the article [9] suggested that there should be an explicit formula for $\Delta(2k+c,k+c)$ with any fixed integer c. In Sections 3 and 4 of the current article, the explicit formulas for all $\Delta(2k+c,k+c)$ with an arbitrary, not necessarily fixed, $c > 0$ are derived. In Section 5, the question regarding the behavior of $\Delta(n,t)$ for small t, as suggested in [9], is revisited.

2. Remark

This article was written after an e-mail exchange with the authors of [3–5], and the present author would like to thank them for a fruitful, helpful, and detailed discussion. In particular, they kindly sent to the present author a draft of the article [4] with the proposed computation of $\Delta(n,t)$ in the range $n \leqslant 2t$ before the present author returned to the work on this topic in December 2021; so, the current paper does not claim the novelty of the result given in Section 3, at least in what concerns its formulation. However, the present author believes that his proof, as presented in Section 4, has the advantage of being very simple, and hence it might be helpful for some readers. In addition, as the first draft of the current paper was completed in 2021 [10], the argument presented in Section 4 might have formed the earliest unconditional corresponding proof because the technique developed in [4] has relied on the earlier work [3], in which the demonstrations of the relevant results required correction [5].

3. The recurrence relation for $n \leqslant 2t$ and its discussion

One of the main results of this note is the following.

Theorem 3.1. *The recurrence relation* $\Delta(n,t) = u_1\,\Delta(n-1,t) + u_2\,\Delta(n-2,t) + \ldots + u_{10}\,\Delta(n-10,t)$ *holds with*

$$(u_1,\ldots,u_{10}) = (2,-3,4,-2,0,2,-4,3,-2,1)$$

for all integers $t \geqslant 15$ *and* $n \in \{t+15, t+16, \ldots, 2t-1, 2t\}$.

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The characteristic polynomial of the relation given in Theorem 3.1 is

$$-\tau^{10}+2\tau^{9}-3\tau^{8}+4\tau^{7}-2\tau^{6}+2\tau^{4}-4\tau^{3}+3\tau^{2}-2\tau+1. \tag{1}$$

Since the explicit formulas for the initial values

$$\Delta(t+1,t),\ \Delta(t+2,t),\ \ldots,\ \Delta(t+14,t) \tag{2}$$

are known from [9], the standard methods of solving linear recurrences yield an explicit formula for $\Delta(n,t)$ for all possible integer pairs (n,t) with $n \leqslant 2t$, which realizes the suggestion proposed in [9]. Moreover, since the sequences (2) are periodic [9] and since the polynomial (1) splits as $(1-\tau)^3(1+\tau^2)^3(1+\tau)$, the absolute values $|\Delta(n,t)|$ remain bounded by an explicit degree-two polynomial of $(n-t)$ in the same range $n \leqslant 2t$.

4. The proof of Theorem 3.1

The approach of this section requires some further auxiliary notation.

Definition 4.1. *Assuming that $t \geqslant 2$, $n \geqslant t+2$ and $i,j \in \{1,2\}$, we declare that*

(i) $\Delta_i(n,t)$ is $(-1)^t$ multiplied by the determinant obtained from $\Delta(n,t)$ by removing the i-th row and $(t+i)$-th column,

(ii) $\Delta_{ij}(n,t)$ is the determinant obtained from $\Delta(n,t)$ by removing the i-th row, $(n+1-t-j)$-th row, $(t+i)$-th column, and $(n+1-j)$-th column, where it is additionally assumed that $n \geqslant t+4$.

We remark that the assumption $n \geqslant t+4$ is taken in (ii) to ensure that the corresponding value $\Delta_{ij}(n,t)$ is indeed well defined. In order to proceed, we need one observation and a further result similar to Claim 7 in [9].

Definition 4.2. *Let $Q(n)$ be the $n\times n$ matrix with ones at the positions (i,j) with $j-i \in \{-2,-1,0,1\}$ and zeros everywhere else, and let $Q'(n)$ be the matrix obtained from $Q(n)$ by replacing the $(3,1)$ entry by a zero whenever $n \geqslant 3$.*

Claim 4.1 (see Claim 3 in [9])**.** *The sequences $\det Q(n)$ and $\det Q'(n)$ have period four.*

Claim 4.2. *One has $\Delta_{ij}(n,t) = \Delta_{ij}(n-4,t)$ for all $n \in \{t+8,\ldots,2t\}$.*

Proof. As in [9, Claim 7], whenever $m \leqslant 2t$, the determinant $\Delta_{ij}(m,t)$ can be written as

$$\left(\begin{array}{c|c|c} M_1 & * & * \\ \hline O & U & * \\ \hline O & O & M_2 \end{array}\right)$$

in which U is a unitriangular square block of the order $2t-m+i+j$, the $*$'s stand for matrices that do not need to be specified, and (M_1,M_2) are square matrices of the orders $m-t-j-1$ and $m-t-i-1$, which correspond to

$$Q(m-t-j-1) \quad \text{and} \quad Q(m-t-i-1), \tag{3}$$

respectively, in the case $(i,j)=1$. Also, a replacement of either i or j by 2 leads to the change of the determinant of the corresponding matrix in (3) as if Q was replaced by Q', so the result follows from Claim 4.1. □

We proceed with the proof of Theorem 3.1. In fact, the expansion of $\Delta(n,t)$ along the first two rows gives

$$\Delta(n,t) = \Delta(n-1,t)-\Delta(n-2,t)+\Delta(n-3,t)+\Delta_2(n,t)+\Delta_1(n,t)-\Delta_1(n-1,t)-\det Q(n-t-2), \tag{4}$$

provided that $n \geqslant t+3$. Similarly, the expansion along the last two columns allows us to write

$$\Delta_i(n,t) = \Delta_i(n-1,t)-\Delta_i(n-2,t)+\Delta_i(n-3,t)+\Delta_{2i}(n,t)+\Delta_{1i}(n,t)-\Delta_{1i}(n-1,t), \tag{5}$$

for any $i \in \{1,2\}$ and $n \geqslant t+5$. In view of Claim 4.2, the formula (5) gives

$$u_0\,\Delta_i(n,t)+u_1\,\Delta_i(n-1,t)+\ldots+u_7\,\Delta_i(n-7,t)=0 \tag{6}$$

with $(u_0,\ldots,u_7)=(1,-1,1,-1,-1,1,-1,1)$ and $n \geqslant t+9$; so, a comparison of (4) and (6) proves Theorem 3.1.

5. On the value of $\Delta(n,t)$ for small t

As explained in [9], for general (n,t), the computation of $\Delta(n,t)$ should be much harder than the case $n \leqslant 2t$ in the previous section. As it turns out, the case when the value of t is fixed does also allow a simple explicit formula for $\Delta(n,t)$, and its derivation can be done with the application of Theorem 2 in [12], which is the result that the author learned in the work [6] together with its applications towards computing $\Delta(n,t)$ and other similar determinants. Indeed, Theorem 2 in [12] guarantees that $\Delta(n,\tau)$ satisfies a linear recurrence of the order $(\tau+1)(\tau+2)/2$ for any fixed τ; and, in particular, the linear recurrence corresponding to $\Delta(n,4)$ turns out to have the characteristic polynomial

$$1 - x^2 - 2x^3 - x^6 - 3x^7 + x^8 + x^9 + 2x^{10} - x^{11} + 2x^{12} - x^{13} + x^{14} - x^{15}$$

in which the root with the largest absolute value $\rho_4 \approx 1.382$ is real, and hence $|\Delta(n,4)|$ is asymptotically equal to $(\rho_4)^n$ times a constant. Several further experiments give the largest absolute values $(\rho_2, \rho_3, \ldots, \rho_{11}, \rho_{12})$ equal to

$$(1.000,\ 1.371,\ 1.382,\ 1.379,\ 1.189,\ 1.283,\ 1.274,\ 1.280,\ 1.186,\ 1.228,\ 1.218)$$

when rounded to the three digits after the decimal point. In particular, we get $\rho_6 < \min\{\rho_5, \rho_7\}$. But, since the values ρ_5 and ρ_7 are realized by pairs of different complex conjugate eigenvalues rather than by real numbers, we rather obtain

$$\Delta(n,5) = A_5 \cdot (\rho_5)^n \cdot \cos(B_5 \cdot n + C_5) + O((\rho_5 - \varepsilon_5)^n) \quad \text{and} \quad \Delta(n,7) = A_7 \cdot (\rho_7)^n \cdot \cos(B_7 \cdot n + C_7) + O((\rho_7 - \varepsilon_7)^n)$$

with some real $(A_5, B_5, C_5, A_7, B_7, C_7)$ and positive $(\varepsilon_5, \varepsilon_7)$. Therefore, the *equidistribution theorem* explains the pattern

$$|\Delta(n+6,6)| < \min\{|\Delta(n+5,5)|, |\Delta(n+7,7)|\} \tag{7}$$

for all n except possibly a set of zero density, which is similar to Conjecture 12 in [9] that states the same inequality (7) but for every $n \geqslant 9$. The exclusion of a possibility of sporadic counterexamples seems to be a much harder task that might require some extensive calculations and advanced number theory, see also [11]. In a similar way, Conjecture 13 in [9] states that

$$|\Delta(n+10,10)| < \min\{|\Delta(n+9,9)|, |\Delta(n+11,11)|\}$$

for all $n \geqslant 95$; and, similarly, we get this conclusion for all positive integers n except possibly a set of zero density by the inequality $\rho_{10} < \min\{\rho_9, \rho_{11}\}$ above. Concerning Conjecture 11 in [9], which states that

$$|\Delta(n+4,4)| > |\Delta(n+m,m)| \tag{8}$$

for all $m \neq 4$ and $n \geqslant 255$, the current methods may allow its full solution; but, in order to keep the presentation concise, we do not go further than giving an explanation of how to reduce the problem to a finite computational task.

Claim 5.1. *One can find an integer $n_1 > 0$ such that (8) is valid if $n > n_1$ and $m > n$.*

Proof. As explained in Section 3, the value $|\Delta(n+m,m)|$ is bounded by an explicit degree-two polynomial of n on this range; while, as noted at the beginning of this section, the growth of $|\Delta(n+4,4)|$ is exponential. □

Claim 5.2. *One can find an integer $n_2 > 0$ such that (8) is valid if $n > n_2$ and $128 \leqslant m \leqslant n$.*

Proof. Let $T(k,a,b)$ be the $k \times k$ matrix in which, for all i and j, the (i,j) entry equals 1 if $i = j$, equals a if $j - i = b$, and equals 0 otherwise. This gives $\det T(k,a,b) = 1$. Taking the right multiplication of the matrix of $\Delta(n+m,m)$ by

$$T(n+m,-1,1) \cdot T\left(n+m,1,2^2\right) \cdot T\left(n+m,1,2^3\right) \cdot \ldots \cdot T\left(n+m,1,2^6\right),$$

we get another $(n+m) \times (n+m)$ matrix $H(n,m)$ of the determinant $\Delta(n+m,m)$ with the following properties:

(i) $H_{j+2,j} = 1$ for all $j \in \{1, \ldots, n+m-2\}$,

(ii) the sum of the absolute values of all the entries in any row is at most 128,

(iii) if one has $H_{ij} \neq 0$ with $i > 2$ and $i \neq j+2$, then $j - i \geqslant 126$.

Since any nonzero summand in the expression of $\det H(n,m)$ such as

$$H_{1,\,\sigma(1)}\cdot\ldots\cdot H_{n+m,\,\sigma(n+m)}$$

should obviously satisfy $(1-\sigma(1))+\ldots+(n+m-\sigma(n+m))=0$, it takes at most $(n+m)/64$ entries as in (iii). Hence, one gets

$$\det H(n,m)\leqslant 128^2\cdot\binom{128(n+m)}{\lceil (n+m)/64\rceil}\leqslant 128^2\cdot\binom{256n}{\lceil n/32\rceil},$$

which does not exceed $(1.37)^n$ for any sufficiently large n. □

Therefore, in order to find an explicit upper bound on the numbers in the pair (n,m) for which the inequality (8) can fail, one needs to compute the explicit formulas for $\Delta(n,j)$ with any fixed $j\in\{2,3\}\cup\{5,6,\ldots,127\}$ (as mentioned at the beginning of this section), provided that the condition $\rho_j<\rho_4$ is valid for all such j. Several initial values, as computed above, give no reason to expect that this condition may fail; but, even if it turns out to fail, its failure would still imply a resolution of Conjecture 11 in [9], in the negative. Therefore, one can reduce the resolution of this conjecture to a finite computational task (which does not look fully unreasonable, but, probably, it still falls short of the abilities of current computers), and a more careful application of the technique presented in this article may reduce the required amount of computation to a feasible one.

6. Acknowledgments

The author is grateful to the reviewers for their interesting comments. In fact, a reviewer suggested that the results of either Claim 5.2 or the forthcoming considerations can be improved; the author agrees with this statement. As discussed in Section 5, the results presented in this article reduce the resolution of [9, Conjecture 11] to a finite computational task, and the author believes that the ideas of Section 5 are more instructive than a potential full solution to [9, Conjecture 11] along the lines presented here, which might require expanding an article by an order of magnitude of more.

References

[1] Y. Amanbek, Z. Du, Y. Erlangga, C. M. da Fonseca, B. Kurmanbek, A. Pereira, Explicit determinantal formula for a class of banded matrices, *Open Math.* **18** (2020) 1227–1229.

[2] M. Anđelić, C. M. da Fonseca, Some determinantal considerations for pentadiagonal matrices, *Linear Multilinear Algebra* **69** (2021) 3121–3129.

[3] Z. Du, C. M. da Fonseca, A periodic determinantal property for $(0,1)$ double banded matrices, *Linear Multilinear Algebra* **70** (2022) 5316–5328.

[4] Z. Du, C. M. da Fonseca, The determinants of certain double banded $(0,1)$ Toeplitz matrices, *Miskolc Math. Notes* **24** (2023) 1307–1315.

[5] Z. Du, C. M. da Fonseca, The periodic determinantal property for $(0,1)$ double banded matrices: a correction and some comments, *Linear Multilinear Algebra* **72** (2024) 1254–1258.

[6] Z. Du, C. M. da Fonseca, A. Pereira, On determinantal recurrence relations of banded matrices, *Kuwait J. Sci.* **49** (2022) 1–9.

[7] J. Feng, H. Fan, A study on the determinants of some Hessenberg Toeplitz Bohemians, Preprint, (2020), DOI: 10.20944/preprints202008.0272.v1.

[8] B. Kurmanbek, Y. Amanbek, Y. Erlangga, A proof of Anđelić–Fonseca conjectures on the determinant of some Toeplitz matrices and their generalization, *Linear Multilinear Algebra* **70** (2022) 1563–1570.

[9] Y. Shitov, The determinants of certain $(0,1)$ Toeplitz matrices, *Linear Algebra Appl.* **618** (2021) 150–157.

[10] Y. Shitov, A further remark on $(0,1)$ Toeplitz determinants, Preprint, (2021), Available at www.researchgate.net/publication/357457503.

[11] C. Stuart, An inequality involving $\sin(n)$, *Amer. Math. Monthly* **125** (2018) 173–174.

[12] H. Zakrajšek, M. Petkovšek, Pascal-like determinants are recursive, *Adv. Appl. Math.* **33** (2004) 431–450.