Discrete Mathematics Letters
www.dmlett.com

Discrete Math. Lett. **14** (2024) 58–65
DOI: 10.47443/dml.2024.132

Research Article

Dynamical structure of metric and linear self-maps on combinatorial trees

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(Received: 19 August 2024. Received in revised form: 24 September 2024. Accepted: 30 September 2024. Published online: 11 October 2024.)

Abstract

The dynamical structure of metric and linear self-maps on combinatorial trees is described. Specifically, the following question is addressed: given a map from a finite set to itself, under what conditions there exists a tree on this set such that the given map is either a metric or a linear map on this tree? The author proves that a necessary and sufficient condition for this is that the map has either a fixed point or a periodic point with period two, in which case all its periodic points must have even periods. The dynamical structure of tree automorphisms and endomorphisms is also described in a similar manner.

Keywords: trees; periodic points; graph maps; metric maps; linear maps; Markov graphs.

2020 Mathematics Subject Classification: 05C05, 37E25, 05C20.

1. Introduction

Let $f : [0,1] \to [0,1]$ be a continuous map from the unit interval to itself, and $x \in [0,1]$ be its periodic point of period n. Then the orbit $\mathrm{orb}_f(x) = \{x, f(x), \dots, f^{n-1}(x)\}$ is a finite set, with the restriction f to $\mathrm{orb}_f(x)$ being a cyclic permutation. One way to encode the information on other possible periodic points for f in this setting is to consider the so-called periodic digraph Γ. This digraph has the vertex set $V(\Gamma) = \{1, \dots, n-1\}$ and the arc set $A(\Gamma) = \{(i,j) : \min\{f(x_i), f(x_{i+1}) \le j < \max\{f(x_i), f(x_{i+1})\}\}$. The idea behind this construction is that the vertices of Γ correspond to the minimal intervals $[x_i, x_{i+1}]$, and the existence of an arc $i \to j$ indicates that the interval $[x_i, x_{i+1}]$ "covers" the interval $[x_j, x_{j+1}]$ under f, i.e., $[x_j, x_{j+1}] \subset f([x_i, x_{i+1}])$. Using this combinatorial approach, one can obtain an elegant proof of the famous Sharkovsky theorem, as shown in [4, 14].

From a purely combinatorial point of view, the construction of periodic digraphs stems from the n-vertex path and its cyclic permutations. Naturally, it can be extended to a broader class of combinatorial trees and their arbitrary vertex self-maps. The corresponding digraphs are called Markov graphs, and they can be used to obtain analogues of the Sharkovsky theorem for self-maps on topological trees [1].

In this paper, we consider metric and linear self-maps on trees. Metric maps (also known as non-expanding, or 1-Lipschitz maps) provide a natural generalization of graph homomorphisms. In fact, they can be defined precisely as homomorphisms between *reflexive* graphs. On the other hand, linear maps are those that "preserve" metric intervals between pairs of vertices. For the properties of metric maps between general graphs we refer to the paper [16]. The papers [10, 11] are devoted to the study of linear and metric self-maps on trees by the means of Markov graphs. We also note that linear maps between median graphs (in particular, trees) can be characterized as maps which preserve medians for all triplets of vertices [12].

Our main result is Theorem 3.1, which completely describes the dynamical structure of metric and linear self-maps on trees. Subsequently, Corollary 3.1 and Proposition 3.2 provide answers to similar questions: when a given self-map of a finite set V is an isomorphism (respectively, an endomorphism) of some tree on V.

A similar problem for lattices and their (anti-)endomorphisms was considered in [3, 15]. For the results on underlying dynamical structure of endomorphisms and automorphisms of certain abelian groups see [6, 7]. Finally, we note that a set V together with a self-map $f : V \to V$ also can be viewed as an algebraic object (the so-called monounary algebra) or as a topological object (the so-called primal space, or a functional Alexandroff space). For an extensive literature on monounary algebras, we refer to the monograph [5], and for more information on primal spaces and functional Alexandroff spaces we refer to the foundational papers in this area [2, 13].

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2. Preliminaries

Graphs and digraphs

In this paper, by a *graph*, we mean a finite undirected simple graph. For convenience, the edges in graphs will be denoted simply as uv. A graph is *connected* provided any pair of its vertices can be joined by a path. The vertex set $V(G)$ of a connected graph G possesses a natural metric d_G, where $d_G(u,v)$ equals the length (i.e., number of edges) of a shortest path between u and v. The *diameter* of a connected graph G is the value $\mathrm{diam}(G) = \max\{d_G(u,v) : u, v \in V(G)\}$. The diameter $\mathrm{diam}(A)$ of a set of vertices $A \subset V(G)$ is then defined as the diameter of the corresponding induced subgraph $G[A]$.

For a pair of vertices $u, v \in V(G)$ in a connected graph G, the set $[u,v]_G = \{x \in V(G) : d_G(u,x) + d_G(x,v) = d_G(u,v)\}$ is called the *metric interval* between u and v. In other words, $[u,v]_G$ consists of all vertices that lie on shortest paths between u, v. A set of vertices $S \subset V(G)$ is called *convex* if $[u,v]_G \subset S$ for all $u, v \in S$. The *convex hull* $\mathrm{Conv}_G(A)$ of a set $A \subset V(G)$ is the smallest convex set containing A. It is clear that $\mathrm{Conv}_G(A)$ equals the intersection of all convex sets S with $A \subset S$.

For a triple of vertices $u, v, w \in V(G)$, let $M_G(u,v,w) = [u,v]_G \cap [u,w]_G \cap [v,w]_G$ define their *median set*. A connected graph G is called *median* if $|M_G(u,v,w)| = 1$ for all $u, v, w \in V(G)$. In this case, the unique vertex in $M_G(u,v,w)$ is called the *median* of the triple u, v, w, and is commonly denoted by $m_G(u,v,w)$.

For an edge $uv \in E(G)$, by $W_G(u,v) = \{x \in V(G) : d_G(u,x) < d_G(v,x)\}$ and $W_G(v,u) = \{x \in V(G) : d_G(v,x) < d_G(u,x)\}$ we denote the corresponding *half-spaces*.

A set of vertices $A \subset V(G)$ is called *Chebyshev* if for every $x \in V(G)$ there exists a unique vertex $a_x \in A$ which minimizes the distance from x to A (i.e., a_x is a unique vertex from A with $d_G(x, a_x) = d_G(x, A) := \min\{d_G(x,a) : a \in A\}$). The corresponding vertex a_x is denoted by $\mathrm{pr}_A(x)$ and is called the *projection* of x onto A.

A *tree* is a connected graph without cycles. Prominent examples of trees include paths and stars. Note also that any tree is a median graph. A vertex of degree 1 in a tree is called a *leaf*. By $\mathrm{Leaf}(X)$ we denote the set of leaves in a tree X.

For a finite set V, by $\mathrm{Tr}(V)$ we denote the collection of all trees X with $V(X) = V$.

Aside from graphs, in this paper we also will be dealing with digraphs. In the next section, we define the so-called Markov graphs for self-maps of trees. For now, note that our digraphs are finite, simple, but they can contain loops as well as cycles of length 2.

For a digraph D, its *converse digraph* D^{co} is obtained from D by reversing arc orientations of D. The *out-degree* of a vertex u in a digraph D is the number $d_D^+(u) = |\{v \in V(D) : (u,v) \in A(D)\}|$. A digraph D is called *partial functional* if $d_D^+(u) \le 1$ for all $u \in V(D)$.

Maps between graphs

Let V be a finite set without any structure. By $\mathcal{T}(V)$ we denote the class of all self-maps of V, i.e., maps of the form $\sigma : V \to V$. The identity map on V is denoted by id_V. The image of a given map $\sigma \in \mathcal{T}(V)$ will be denoted as $\mathrm{Im}\,\sigma$.

An element $x \in V$ is called a *fixed point* of a map $\sigma \in \mathcal{T}(V)$ provided $\sigma(x) = x$. The set of all fixed points of σ is denoted by $\mathrm{fix}\,\sigma$. An element $x \in V$ is called a *σ-periodic point* if there exists $k \in \mathbb{N}$ with $\sigma^k(x) = x$. The smallest such a k is called the *period* of x. Hence, fixed points are exactly σ-periodic points of period 1. By $\mathrm{per}\,\sigma$ we denote the set of all σ-periodic points. The set $\mathrm{orb}_\sigma(x) = \{x, \sigma(x), \dots, \sigma^n(x), \dots\}$ is called the *orbit* of x. A set $A \subset V$ is called *σ-invariant* if $\sigma(A) \subset A$. For example, the orbit $\mathrm{orb}_\sigma(x)$ of any element $x \in V$ is always a σ-invariant set. Note also that every non-empty σ-invariant set contains at least one σ-periodic point.

Now let X be a tree and $\sigma : V(X) \to V(X)$ be its vertex self-map. The corresponding *Markov graph* is a digraph $\Gamma = \Gamma(X, \sigma)$ with the vertex set $V(\Gamma) = E(X)$ and the arc set $A(\Gamma) = \{(uv, xy) : x, y \in [\sigma(u), \sigma(v)]_X\}$. In this digraph, the vertices of Γ represent the edges of X, and there is an arc $uv \to xy$ provided the edge uv "covers" xy under the map σ.

Example 2.1. *Let X be a tree with $V(X) = \{1, \dots, 7\}$ and $E(X) = \{12, 23, 34, 45, 26, 37\}$. Also, consider the map* $\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 4 & 1 & 3 & 6 & 2 & 4 & 2 \end{pmatrix}$. *Then the corresponding Markov graph $\Gamma(X, \sigma)$ is depicted in Figure 2.1.*

The following proposition provides simple (but useful) bounds on the size of Markov graphs.

Proposition 2.1. [8] *Let X be an n-vertex tree and $\sigma : V(X) \to V(X)$ be its self-map. Then*

$$|\mathrm{Im}\,\sigma| - 1 \le |A(\Gamma(X, \sigma))| \le (n-1) \cdot \mathrm{diam}(\mathrm{Im}\,\sigma).$$

In this paper, we consider different classes of maps on trees. A map $\sigma : V(G) \to V(H)$ between the vertex sets of two connected graphs G and H is called

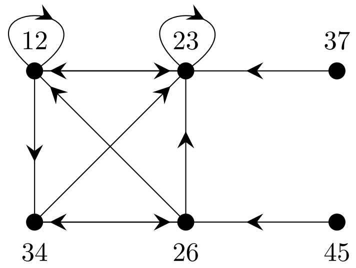

Figure 2.1: The Markov graph $\Gamma(X,\sigma)$ for the pair (X,σ) from Example 2.1.

- a *homomorphism* if $f(u)f(v) \in E(H)$ for all $uv \in E(G)$;
- an *isomorphism* if σ is a bijective homomorphism and f^{-1} is also a homomorphism;
- a *metric map*, if $d_H(\sigma(u),\sigma(v)) \le d_G(u,v)$ for all $u,v \in V(G)$;
- a *linear map*, if $f([u,v]_G) \subset [f(u),f(v)]_H$ for all $u,v \in V(G)$.

A homomorphism (isomorphism) of a graph G to itself is called an *endomorphism* (*automorphism*) of G.

It is straightforward to prove that a map $\sigma : V(G) \to V(H)$ is metric if and only if $d_G(\sigma(u),\sigma(v)) \le 1$ for all edges $uv \in E(X)$ (see [10]). Hence, each homomorphism is a metric map. Similarly, linear maps between median graphs (in particular, between trees) can be characterized as maps which preserve medians.

Proposition 2.2. [10, 12] *Let $f : V(G) \to V(H)$ be a map between two median graphs G and H. Then f is linear if and only if $f(m_G(u,v,w)) = m_H(f(u),f(v),f(w))$ for all triples of vertices $u,v,w \in V(G)$.*

For trees, there is a notable duality between metric and linear self-maps, as suggested by the following theorem.

Theorem 2.1. [10] *Let X be a tree and $\sigma : V(X) \to V(X)$ be a map. Then*

1. *σ is metric if and only if $\Gamma(X,\sigma)$ is partial functional;*
2. *σ is linear if and only if the converse digraph $(\Gamma(X,\sigma))^{co}$ is partial functional.*

Moreover, one can characterize the maps which achieve the lower bound from Proposition 2.1.

Theorem 2.2. [10] *Let X be a tree and $\sigma : V(X) \to V(X)$ be its map. Then $|A(\Gamma(X,\sigma))| = |\operatorname{Im}\sigma| - 1$ if and only if σ is a linear metric map.*

One example of a tree self-map which is linear and metric at the same time is provided by the projection map $\mathrm{pr}_A : V(X) \to V(X)$ on a Chebyshev (equivalently, connected) set of vertices $A \subset V(X)$. Notably, projections on connected sets with at least 3 elements in trees can be characterized in terms of the corresponding Markov graphs (see [8]).

As mentioned in the previous section, our digraphs can have loops. Specifically, the Markov graph $\Gamma(X,\sigma)$ has a loop at the vertex $uv \in E(X)$ if and only if $u,v \in [\sigma(u),\sigma(v)]_X$. However, there are two types of loops in Markov graphs. Namely, an edge $uv \in E(X)$ is called *σ-positive* if $\sigma(u) \in W_G(u,v)$ and $\sigma(v) \in W_G(v,u)$. Similarly, an edge $uv \in E(X)$ is called *σ-negative* if $\sigma(u) \in W_G(v,u)$ and $\sigma(v) \in W_G(u,v)$. By $p(X,\sigma)$ and $n(X,\sigma)$ we denote the numbers of σ-positive and σ-negative edges in X, respectively. These numbers are related by the following equality.

Theorem 2.3. [9] *For any tree X and a map $\sigma : V(X) \to V(X)$, it holds $n(X,\sigma) + |\operatorname{fix}\sigma| = p(X,\sigma) + 1$.*

For metric and linear maps, we can say even more.

Proposition 2.3. [11] *Let $\sigma : V(X) \to V(X)$ be a linear or a metric map on a tree X. Then $n(X,\sigma) \le 1$. In addition, the equality $n(X,\sigma) = 1$ necessarily implies $p(X,\sigma) = 0$.*

3. Main results

Before proving the main result of the paper, we note that it is trivial to describe self-maps on finite sets V which are metric (equivalently, linear) for *all* trees on V.

Proposition 3.1. *Let $n = |V| \geq 3$. For a map $\sigma \in \mathcal{T}(V)$ the following statements are equivalent:*

1. *σ is constant or $\sigma = \mathrm{id}_V$;*
2. *for every tree $X \in \mathrm{Tr}(V)$ the map σ is metric on X;*
3. *for every tree $X \in \mathrm{Tr}(V)$ the map σ is linear on X.*

Proof. The implications $1 \Rightarrow 2$ and $1 \Rightarrow 3$ are trivial.

To prove the implication $2 \Rightarrow 1$, suppose that $\sigma \neq \mathrm{id}_V$ is a non-constant map. Then there exist vertices $u, v \in V$ with $\sigma(u) \neq u$ and $\sigma(v) \neq \sigma(u)$. In particular, $u \neq v$.

If $\sigma(u) \neq v$ and $\sigma(v) \neq u$, then consider any path $X \in \mathrm{Tr}(V)$ with $uv \in E(X)$ and $\mathrm{Leaf}(X) = \{\sigma(u), \sigma(v)\}$. The inequality $d^+_{\Gamma(X,\sigma)}(uv) = |V(\Gamma(X,\sigma))| = |E(X)| = n-1 \geq 2$ implies that σ is not a metric map on X. If $\sigma(u) = v$ and $\sigma(v) \neq u$ (similarly, if $\sigma(u) \neq v$ and $\sigma(v) = u$), then for any path $X \in \mathrm{Tr}(V)$ with $uv \in E(X)$ and $\mathrm{Leaf}(X) = \{v, \sigma(v)\}$, the map σ is not metric on X.

Finally, consider the case where $\sigma(u) = v$ and $\sigma(v) = u$. Since $n \geq 3$, the set $V \setminus \{u, v\}$ is nonempty. Suppose there exists an element $w \in V \setminus \{u, v\}$ such that $\sigma(w) \neq w$. Without loss of generality, assume that $\sigma(w) \neq v$ (a similar argument applies if $\sigma(w) \neq u$). Then, for any path $X \in \mathrm{Tr}(V)$ with $uw \in E(X)$ and $\mathrm{Leaf}(X) = \{v, \sigma(w)\}$, the map σ is not metric on X. Otherwise, let $V \setminus \{u, v\} \subset \mathrm{fix}\,\sigma$. If $n \geq 4$, then for any path $X \in \mathrm{Tr}(V)$ with $\mathrm{Leaf}(X) = \{u, v\}$, the map σ is not metric on X. Thus, we must consider the case with $n = 3$ and $V = \{u, v, w\}$. In this case, for the path $X \in \mathrm{Tr}(V)$ with $E(X) = \{uv, vw\}$ it holds $d^+_{\Gamma(X,\sigma)}(vw) = 2$ which again implies that σ is not metric on X.

Now we prove the implication $3 \Rightarrow 1$. Let σ be linear on any tree $X \in \mathrm{Tr}(V)$. Suppose that σ is not a permutation. Then there exists $u \in V$ with $|\sigma^{-1}(u)| \geq 2$. Fix two distinct elements $v, w \in \sigma^{-1}(u)$. Fix an arbitrary $x \in V$ and consider the star $X \in \mathrm{Tr}(V)$ centered at x. Since σ is linear on X and $x \in [v, w]_X$, we have $\sigma(x) \in [\sigma(v), \sigma(w)]_X = \{u\}$, i.e. $\sigma(x) = u$. Hence, in this case, σ must be a constant map. Further, suppose that σ is a non-identity permutation. Then there exists $u \in V$ with $\sigma(u) \neq u$. Similarly, consider the star $X \in \mathrm{Tr}(V)$ centered at u. Since $n \geq 3$, there is an element $v \in V \setminus \{u, \sigma(u)\}$. Clearly, $\sigma(u) \in \mathrm{Leaf}(X)$ as well as $\sigma^2(u) \neq \sigma(u)$ and $\sigma(v) \neq \sigma(u)$. This means that $\sigma(u) \notin [\sigma^2(u), \sigma(v)]_X$ for $u \in [\sigma(u), v]_X$, which leads to a contradiction. □

Now, we are ready to state and prove the main theorem of the paper, which completely characterizes the dynamical structure of metric and linear maps on trees.

Theorem 3.1. *For a map $\sigma \in \mathcal{T}(V)$ the following statements are equivalent:*

1. *σ has a fixed point, or there is a σ-periodic point with period two, in which case all σ-periodic points have even periods;*
2. *there exists a tree $X \in \mathrm{Tr}(V)$ such that σ is metric on X;*
3. *there exists a tree $X \in \mathrm{Tr}(V)$ such that σ is linear on X;*
4. $\min\limits_{X \in \mathrm{Tr}(V)} |A(\Gamma(X,\sigma))| = |\mathrm{Im}\,\sigma| - 1.$

Proof. From Theorem 2.2, the implications $4 \Rightarrow 2$ and $4 \Rightarrow 3$ are evident.

Let us now prove the implication $2 \Rightarrow 1$. Suppose that σ is a metric map on a tree $X \in \mathrm{Tr}(V)$ and assume that σ has no fixed points. By Theorem 2.3, we have $n(X,\sigma) = p(X,\sigma) + 1 \geq 1$. Combining this with Proposition 2.3, we can conclude that $n(X,\sigma) = 1$. Let $e = uv \in E(X)$ be the unique σ-negative edge in X. Since σ is metric on X, we have $\sigma(u) = v$ and $\sigma(v) = u$. Therefore, u is a σ-periodic point with period two.

Further, for each σ-periodic point $x \in V \setminus \{u, v\}$, consider the function $f : \mathrm{orb}_\sigma(x) \to \mathbb{N}$ defined as $f(y) = d_X(y, \mathrm{pr}_e(y))$, where $y \in \mathrm{orb}_\sigma(x)$. For all $y \in \mathrm{orb}_\sigma(x)$ with $\mathrm{pr}_e(y) \neq \mathrm{pr}_e(\sigma(y))$ (this means that y and $\sigma(y)$ lie in different half-spaces generated by the edge uv) we have

$$f(\sigma(y)) = d_X(\sigma(y), \mathrm{pr}_e(\sigma(y))) = d_X(\sigma(y), \sigma(\mathrm{pr}_e(y))) \leq d_X(y, \mathrm{pr}_e(y)) = f(y).$$

Similarly, for all $y \in \mathrm{orb}_\sigma(x)$ with $\mathrm{pr}_e(y) = \mathrm{pr}_e(\sigma(y))$, we obtain the inequality $f(\sigma(y)) \leq f(y) - 1$. In other words, $f(\sigma(y)) \leq f(y)$ for all $y \in \mathrm{orb}_\sigma(x)$. Since the restriction of σ to $\mathrm{orb}_\sigma(x)$ is a cyclic permutation, we can conclude that f must be a constant function. Therefore, $\mathrm{pr}_e(y) \neq \mathrm{pr}_e(\sigma(y))$ for all $y \in \mathrm{orb}_\sigma(x)$, which implies that x has an even period under σ.

Now we show that $3 \Rightarrow 1$. Suppose there is a tree $X \in \mathrm{Tr}(V)$ such that σ is linear on X, and also suppose σ has no fixed points. Again, by Theorem 2.3 and Proposition 2.3, we obtain $n(X,\sigma) = 1$. Let $e = uv \in E(X)$ be the unique σ-negative edge in X. Now assume $x \in V$ is a σ-periodic point with an odd period. Then $|\mathrm{orb}_\sigma(x) \cap W_X(u,v)| \neq |\mathrm{orb}_\sigma(x) \cap W_X(v,u)|$.

Without loss of generality, we can assume that $|\operatorname{orb}_\sigma(x)\cap W_X(u,v)| > |\operatorname{orb}_\sigma(x)\cap W_X(v,u)|$. This implies the existence of an element $y\in \operatorname{orb}_\sigma(x)\cap W_X(u,v)$ with $\sigma(y)\in W_X(u,v)$. Consequently, $u\in[y,v]_X$, but $\sigma(u)\notin[\sigma(y),\sigma(v)]_X$ as $\sigma(u)\in W_X(v,u)$ and $\sigma(v)\in W_X(u,v)$. This is impossible because σ is linear on X. Therefore, every σ-periodic point has an even period.

Further, let $l(\sigma)$ denote the least period of a σ-periodic point. Since $\operatorname{fix}\sigma=\emptyset$, we have $l(\sigma)\geq 2$. We need to prove that $l(\sigma)=2$. To the contrary, suppose that $l(\sigma)\geq 4$ and consider a σ-periodic point $x\in V$ with period $l(\sigma)$. Consider the set $A=\operatorname{Conv}_X(\operatorname{orb}_\sigma(x))$ and let $X'=X[A]$, $n_1=|\operatorname{Leaf}(X')|$, $n_2=|\{y\in V(X'): d_{X'}(y)=2\}|$ and $n_3=|\{y\in V(X'): d_{X'}(y)\geq 3\}|$. We have the inequality

$$3n_3+2n_2+n_1\leq \sum_{y\in V(X')} d_{X'}(y)=2|E(X')|=2(|V(X')|-1)=2(n_1+n_2+n_3-1),$$

which trivially implies $n_3\leq n_1-2$. Put $m_i=m_X(\sigma^{i-1}(x),\sigma^i(x),\sigma^{i+1}(x))$ for every $1\leq i\leq l(\sigma)$. By Proposition 2.2,

$$\sigma(m_i)=\sigma(m_X(\sigma^{i-1}(x),\sigma^i(x),\sigma^{i+1}(x)))=m_X(\sigma^i(x),\sigma^{i+1}(x),\sigma^{i+2}(x))=m_{i+1}$$

for all $1\leq i\leq l(\sigma)-1$ and $\sigma(m_{l(\sigma)})=m_1$. Thus, the set of medians $M=\{m_i: 1\leq i\leq l(\sigma)\}$ is σ-invariant. Further, it is easy to see that $M\cap\operatorname{Leaf}(X')=\emptyset$. If $d_{X'}(m_i)=2$ for some $1\leq i\leq l(\sigma)$, then $m_i\in\{\sigma^{i-1}(x),\sigma^i(x),\sigma^{i+1}(x)\}$. This implies the equality $M=\operatorname{orb}_\sigma(x)$ which is impossible since $\emptyset\neq\operatorname{Leaf}(X')\subset\operatorname{orb}_\sigma(x)$. Hence, $d_{X'}(m_i)\geq 3$ for all $1\leq i\leq l(\sigma)$ which gives $|M|\leq n_3$. Therefore, there is a σ-periodic point $y\in M$ of period at most $|M|\leq n_3\leq n_1-2\leq|\operatorname{orb}_\sigma(x)|-2=l(\sigma)-2$. This contradicts the minimality of $l(\sigma)$. Thus, $l(\sigma)=2$.

Finally, we prove the implication $1\Rightarrow 4$. Proposition 2.1 asserts that $\min_{X\in\operatorname{Tr}(V)}|A(\Gamma(X,\sigma))|\geq|\operatorname{Im}\sigma|-1$ for all maps $\sigma\in\mathcal{T}(V)$. Hence, for a given map $\sigma\in\mathcal{T}(V)$ that satisfies the first condition, we must construct a tree $X\in\operatorname{Tr}(V)$ on which σ achieves this bound.

First, suppose that σ is a permutation of V. If there is a fixed point $u\in\operatorname{fix}\sigma$, then for the star $X\in\operatorname{Tr}(V)$ centered at u, the map σ is an automorphism of X. In particular, $|A(\Gamma(X,\sigma))|=|E(X)|=|V|-1=|\operatorname{Im}\sigma|-1$. Thus, assume σ has no fixed points. Consider the partition $V=\bigsqcup_{i=1}^k\operatorname{orb}_\sigma(u_i)$ of V into the orbits of σ-periodic points u_i, $1\leq i\leq k$. Without loss of generality, we can assume that $|\operatorname{orb}_\sigma(u_1)|=l(\sigma)=2$.

For each $1\leq i\leq k$, define two sets $A_i=\{\sigma^m(u_i): m\in\mathbb{N}\text{ is odd}\}$ and $B_i=\{\sigma^m(u_i): m\in\mathbb{N}\text{ is even}\}$. Further, consider a graph X on V with the edge set

$$E(X)=\{u_1x: x\in\bigcup_{i=1}^k A_i\}\cup\{\sigma(u_1)x: x\in\bigcup_{i=1}^k B_i\}.$$

One can observe that $X\in\operatorname{Tr}(V)$ (in fact, X is the so-called *bi-star* having central edge $u_1\sigma(u_1)$) and σ is an automorphism of X. In particular, this gives $|A(\Gamma(X,\sigma))|=|\operatorname{Im}\sigma|-1$.

Further, we use induction on $n=|V|\geq 1$. If σ is a permutation, then we are done. Otherwise, $V\setminus\operatorname{Im}\sigma\neq\emptyset$. For every $u\in V\setminus\operatorname{Im}\sigma$ define

$$f(u)=\min\{k\in\mathbb{N}: |\sigma^{-1}(\sigma^k(u))|\geq 2\}.$$

Since V is finite, the map $f: V\setminus\operatorname{Im}\sigma\to\mathbb{N}$ is well-defined. Fix an element $u_0\in V\setminus\operatorname{Im}\sigma$ such that $f(u_0)=\min\{f(u): u\in V\setminus\operatorname{Im}\sigma\}$, and consider the set

$$V'=V\setminus\{\sigma^i(u_0): 0\leq i\leq f(u_0)-1\}.$$

Clearly, the restriction $\sigma'=\sigma|_{V'}$ is a self-map on V'. Also, $\operatorname{Im}\sigma'=\operatorname{Im}\sigma\setminus\{\sigma^i(u_0): 0\leq i\leq f(u_0)-1\}$, which implies $|\operatorname{Im}\sigma'|=|\operatorname{Im}\sigma|-f(u_0)+1$. Additionally, an element $x\in V'$ is a σ'-periodic point if and only if x is a σ-periodic point. Thus, σ' also satisfies the first condition of the theorem.

We have $|V'|<n$, so by induction assumption, there exists a tree $X'\in\operatorname{Tr}(V')$ with $|A(\Gamma(X',\sigma'))|=|\operatorname{Im}\sigma'|-1=|\operatorname{Im}\sigma|-f(u_0)$. Since $|\sigma^{-1}(\sigma^{f(u_0)}(u_0))|\geq 2$, there exists $y\in\sigma^{-1}(\sigma^{f(u_0)}(u_0))\setminus\{\sigma^{f(u_0)-1}(u_0)\}$.

We aim to show that $\sigma^{-(f(u_0)-1)}(y)\cap V'\neq\emptyset$.

First, assume $y\in\operatorname{per}\sigma$ is a σ-periodic point. Then $\sigma^{f(u_0)}(u_0)=\sigma(y)$ is also σ-periodic. Since $\operatorname{orb}_\sigma(y)=\operatorname{orb}_\sigma(\sigma^{f(u_0)}(u_0))$, it is evident that $\operatorname{orb}_\sigma(y)$ cannot share elements with the set $\{\sigma^i(u_0): 0\leq i\leq f(u_0)-1\}$. Trivially, in this case, $\sigma^{-k}(y)\cap\operatorname{orb}_\sigma(y)\neq\emptyset$ for all $k\geq 1$.

Second, suppose y is not σ-periodic. Towards contradiction, assume that

$$g(y):=\max\{k\in\mathbb{N}\cup\{0\}: \sigma^{-k}(y)\neq\emptyset\}<f(u_0)-1.$$

Fix an element $z\in\sigma^{-g(y)}(y)$. Clearly, $z\in V\setminus\operatorname{Im}\sigma$. But then $f(z)\leq g(y)+1<f(u_0)$, which contradicts the minimality of $f(u_0)$. Hence, in this case, $g(y)\geq f(u_0)$, implying $\sigma^{-(f(u_0)-1)}(y)\neq\emptyset$. Moreover, since $y\neq\sigma^{f(u_0)-1}(u_0)$, it holds $\sigma^{-(f(u_0)-1)}(y)\subset V'$.

Thus, we can fix an element $z \in \sigma^{-(f(u_0)-1)}(y) \cap V'$. Further, consider a graph X on V with the edge set

$$E(X) = E(X') \cup \{\sigma^i(z)\sigma^i(u_0) : 0 \le i \le f(u_0) - 1\}.$$

It is easy to see that $X \in \mathrm{Tr}(V)$ is a tree on V since X is obtained from the tree X' by adding several leaf vertices. Moreover,

$$|A(X,\sigma)| = |A(\Gamma(X',\sigma'))| + \sum_{i=0}^{f(u_0)-1} d^+_{\Gamma(X,\sigma)}(\sigma^i(z)\sigma^i(u_0)) = |\operatorname{Im}\sigma| - f(u_0) + \sum_{i=0}^{f(u_0)-1} d_X(\sigma^{i+1}(z), \sigma^{i+1}(u_0)).$$

For $i = f(u_0) - 1$, it holds $d_X(\sigma^{i+1}(z), \sigma^{i+1}(u_0)) = d_X(\sigma^{f(u_0)}(z), \sigma^{f(u_0)}(u_0)) = 0$. And for each $0 \le i \le f(u_0) - 2$, we have $d_X(\sigma^{i+1}(z), \sigma^{i+1}(u_0)) = 1$. Therefore, the size of $\Gamma(X,\sigma)$ equals

$$|A(\Gamma(X,\sigma))| = |\operatorname{Im}\sigma| - f(u_0) + \sum_{i=0}^{f(u_0)-1} d_X(\sigma^{i+1}(z), \sigma^{i+1}(u_0)) = |\operatorname{Im}\sigma| - f(u_0) + f(u_0) - 1 = |\operatorname{Im}\sigma| - 1.$$

□

We illustrate some of the ideas from the proof of Theorem 3.1 in the next example.

Example 3.1. *Consider a set $V = \{1,\dots,9\}$ and its self-map $\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 2 & 1 & 4 & 5 & 6 & 3 & 1 & 4 & 8 \end{pmatrix}$. It is clear that 1 is a σ-periodic point of period two, and all other σ-periodic points have periods two and four. Hence, Theorem 3.1 suggests that there is a tree $X \in \mathrm{Tr}(V)$ with σ being both metric and linear on X. To construct such a tree X, start with the "bijective part" of σ, i.e., the restriction of σ to the set $\operatorname{per}\sigma$ of all σ-periodic points. In our example, this would be the permutation $\sigma' = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 1 & 4 & 5 & 6 & 3 \end{pmatrix}$ which acts on $\operatorname{per}\sigma = \{1,2,3,4,5,6\}$.*

Let us first construct a tree $X' \in \mathrm{Tr}(\operatorname{per}\sigma)$ where σ' is both metric and linear (in fact, an automorphism) on X'. We start by defining the edge set $E(X') = \{12, 13, 15, 24, 26\}$. Next, we take the element $7 \in V \setminus \operatorname{per}\sigma$ and add to X' the new edge 27 (the idea here is that $\sigma(1) = 2$, thus the image of the edge 27 under σ will collapse into a vertex 1). Further, we take $8 \in V \setminus \operatorname{per}\sigma$ and add an edge 38 (again, using the fact that $\sigma(3) = \sigma(8) = 4$). Finally, we add an edge 69 (here we use the fact that $\sigma(6) = 3$ implying that the image of 69 under σ will be a newly added edge 38). Formally, we consider the tree $X \in \mathrm{Tr}(V)$ having $E(X) = \{12, 13, 15, 24, 26, 27, 38, 69\}$ (see Figure 3.1). Then the corresponding Markov graph $\Gamma(X,\sigma)$ has exactly 6 arcs which equals $|\operatorname{Im}\sigma| - 1$. Thus, Theorem 3.1 implies that σ is indeed a metric and a linear map on X.

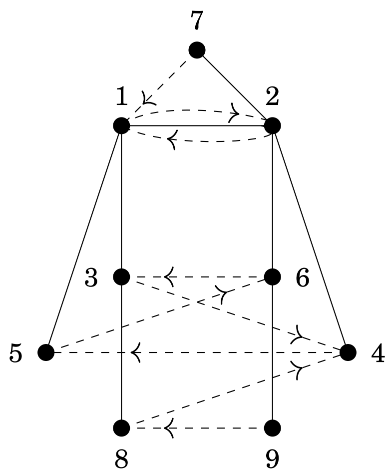

Figure 3.1: The tree X for the map σ from Example 3.1.

Note that a linear endomorphism of a tree must be an automorphism. Indeed, let $\sigma : V(X) \to V(X)$ be simultaneously a linear map and an endomorphism of a tree X. It is not difficult to show that the pre-image $\sigma^{-1}(y)$ of any vertex $y \in \operatorname{Im}\sigma$ is a connected set, implying that $|\sigma^{-1}(y)| = 1$ (as otherwise, $\sigma^{-1}(y)$ would contain an edge, but this contradicts the fact that σ is an endomorphism of X). Thus, σ is injective, implying that σ is bijective by the finitness of X. And the bijective endomorphism of a finite tree is its automorphism (again, because it induces an injective self-map on $E(X)$). We will use this observation in proving the following corollary.

Corollary 3.1. *For a permutation σ of V, there exists a tree $X \in \mathrm{Tr}(V)$ such that σ is an automorphism of X if and only if σ has a fixed point, or there is a σ-periodic point with period two, in which case all σ-periodic points have even periods.*

Proof. Theorem 3.1 immediately implies the necessity of the condition, as an automorphism is both a metric and a linear map. For the sufficiency, Theorem 3.1 ensures that for a map $\sigma \in \mathcal{T}(V)$ with given properties, there exists a tree $X \in \mathrm{Tr}(V)$ such that σ is both metric and linear on X. Since σ is bijective, it is an endomorphism of X. Thus, σ is a linear endomorphism of X. Hence, by the discussion preceding this proposition, σ is an automorphism of X. □

Similarly to Theorem 3.1, we can obtain a complete description of the dynamical structure of tree endomorphisms.

Proposition 3.2. *Let $n = |V| \geq 3$. For a map $\sigma \in \mathcal{T}(V)$, there exists a tree $X \in \mathrm{Tr}(V)$ such that σ is an endomorphism of X if and only if σ satisfies one of the following conditions:*

1. $\mathrm{fix}\,\sigma \neq \emptyset$ *and* $|\mathrm{per}\,\sigma| \geq 2$*;*
2. *there exists a σ-periodic point with period two, all σ-periodic points have even periods, and* $|\mathrm{per}\,\sigma| \geq 3$.

Proof. Necessity. Suppose σ has a fixed point, say $u \in \mathrm{fix}\,\sigma$. Then its neighborhood $N_X(u)$ is a non-empty (as $n \geq 3$) σ-invariant set. Therefore, $N_X(u)$ contains a σ-periodic point different from u. Thus, in this case, $|\mathrm{per}\,\sigma| \geq 2$.

Next, assume that $\mathrm{fix}\,\sigma = \emptyset$. As σ is an endomorphism, it is a metric map. Hence, Theorem 3.1 implies that there is a σ-periodic point with period two, and all σ-periodic points have even periods. Moreover, if $u \in V(X)$ is a σ-periodic point with period two, then the set $(N_X(u) \cup N_X(\sigma(u))) \setminus \{u, \sigma(u)\}$ is non-empty (as $n \geq 3$) and σ-invariant. Hence, σ must have at least one additional periodic point other than $u, \sigma(u)$.

Sufficiency. First, suppose that σ has a fixed point and $|\mathrm{per}\,\sigma| \geq 2$. Let $u_0 \in \mathrm{fix}\,\sigma$ and $v_0 \in \mathrm{per}\,\sigma \setminus \{u_0\}$. Consider the restriction $\sigma_0 = \sigma|_{\mathrm{orb}_\sigma(v_0)}$ of σ to the orbit $\mathrm{orb}_\sigma(v_0)$. It is clear that σ_0 is a cyclic permutation of $\mathrm{orb}_\sigma(v_0)$.

Consider a σ-invariant set $S = \bigcup_{n=1}^{+\infty} \sigma^{-n}(u_0)$ and the map $\kappa : S \to \mathbb{N} \cup \{0\}$, where $\kappa(x) = \min\{k \in \mathbb{N} \cup \{0\} : \sigma^k(x) = u_0\}$. Of course, $\kappa(u_0) = 0$. Now consider a graph X on V with the edge set

$$E(X) = \{u_0 x : x \in V \setminus S\} \cup \{x\sigma_0^{-\kappa(x)}(v_0) : x \in S \setminus \{u_0\}\}.$$

It is easy to see that $X \in \mathrm{Tr}(V)$. We claim that σ is an endomorphism of X. To show this, we must consider two types of edges in X. For the edges of the from $u_0 x \in E(X)$ with $x \notin S$, we have $\sigma(u_0) = u_0$ and $\sigma(x) \notin S$ (as otherwise, $\sigma(x) \in S$ would imply $x \in S$ by the definition of the set S). Thus, $\sigma(u_0)\sigma(x) \in E(X)$ is an edge. Similarly, for the edges of the form $x\sigma_0^{-\kappa(x)}(v_0) \in E(X)$ with $x \in S \setminus \{u_0\}$, we observe that $\kappa(\sigma(x)) = \kappa(x) - 1$. Thus, the vertex

$$\sigma(\sigma_0^{-\kappa(x)}(v_0)) = \sigma_0^{-\kappa(x)+1}(v_0) = \sigma_0^{-(\kappa(x)-1)}(v_0) = \sigma_0^{-\kappa(\sigma(x))}(v_0)$$

is adjacent to the vertex $\sigma(x)$. This proves the proposition in the case where $\mathrm{fix}\,\sigma \neq \emptyset$.

Now assume that σ satisfies the second condition. We follow a similar approach to the proof of the implication $1 \Rightarrow 4$ in Theorem 3.1, but in a slightly modified way. First, consider the partition $\mathrm{per}\,\sigma = \bigsqcup_{i=1}^{k} \mathrm{orb}_\sigma(u_i)$ of $\mathrm{per}\,\sigma$ into the orbits of σ-periodic points u_i, $1 \leq i \leq k$. Without loss of generality, we can assume that $|\mathrm{orb}_\sigma(u_1)| = 2$. Now for $x \in V$, put

$$\kappa(x) = \min\{m \in \mathbb{N} \cup \{0\} : \sigma^m(x) \in \{u_1, \ldots, u_k\}\}.$$

Then we obtain a well-defined map $\kappa : V \to \mathbb{N} \cup \{0\}$.

Now partition the set V into two disjoint sets $A = \{x \in V : \kappa(x) \textit{ is odd}\}$ and $B = \{x \in V : \kappa(x) \textit{ is even}\}$. Consider a graph X on V with the edge set

$$E(X) = \{u_1 x : x \in A\} \cup \{\sigma(u_1)x : x \in B\}.$$

One can easily observe that $X \in \mathrm{Tr}(V)$ is a bi-star. Moreover, σ is an endomorphism of X. Indeed, for an edge of the form $u_1 x \in E(X)$ with $x \in A$, it holds $\kappa(\sigma(x)) = \kappa(x) - 1$. Hence, $\sigma(x) \in B$ implying that $\sigma(u_1)$ is adjacent to $\sigma(x)$. A similar argument works for the edges of the form $\sigma(u_1)x \in E(X)$ with $x \in B$. □

Finally, we provide two examples of maps σ which satisfy each of the two conditions from Proposition 3.2, and construct the corresponding trees X for them.

Example 3.2. *1. Consider a set $V = \{1, \ldots, 9\}$ and its self-map* $\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 1 & 1 & 2 & 2 & 1 & 7 & 8 & 6 & 8 \end{pmatrix}$*. It is evident that 1 is a fixed point for σ, and $\mathrm{per}\,\sigma = \{1, 6, 7, 8\}$. Hence, σ satisfies the first condition from Proposition 3.2, implying that there exists a tree $X \in \mathrm{Tr}(V)$ with σ being an endomorphism on X. Let us construct this X. In the setting of the proof of sufficiency in Proposition 3.2, let $u_0 = 1$ and $v_0 = 6$. Also, $S = \bigcup_{n=1}^{+\infty} \sigma^{-n}(u_0) = \{1, 2, 3, 4, 5\}$. Further, we have $\kappa(1) = 0$, $\kappa(2) = \kappa(5) = 1$, and $\kappa(3) = \kappa(4) = 2$. Thus, the edge set of X is $E(X) = \{16, 17, 18, 19, 28, 58, 37, 47\}$ (see Figure 3.2). One can check by hand that σ is indeed an endomorphism of X.*

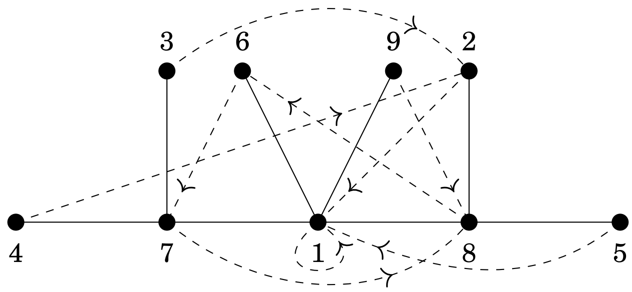

Figure 3.2: The tree X for the first map σ from Example 3.2.

2. *Again, let* $V = \{1, \ldots, 9\}$*, but now* $\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 2 & 1 & 2 & 2 & 4 & 7 & 8 & 9 & 6 \end{pmatrix}$*. It is clear that* 1 *is a* σ*-periodic point with period two, and* $\operatorname{per}\sigma = \{1, 2, 6, 7, 8, 9\}$*. Therefore,* σ *satisfies the second condition from Proposition 3.2. Thus, again, there is a tree* $X \in \operatorname{Tr}(V)$ *with* σ *being an endomorphism on* X*. Let us construct this* X*. In the setting of the proof of sufficiency in Proposition 3.2, let* $u_1 = 1$ *and* $u_2 = 6$*. Also, we have* $\kappa = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 0 & 1 & 2 & 2 & 3 & 0 & 3 & 2 & 1 \end{pmatrix}$*. Thus,* $A = \{x \in V : \kappa(x) \text{ is odd}\} = \{2, 5, 7, 9\}$ *and* $B = \{x \in V : \kappa(x) \text{ is even}\} = \{1, 3, 4, 6, 8\}$*. The corresponding bi-star* X *has the edge set* $E(X) = \{12, 13, 15, 17, 19, 24, 26, 28\}$ *(see Figure 3.3). One can verify by hand that* σ *is indeed an endomorphism of* X*.*

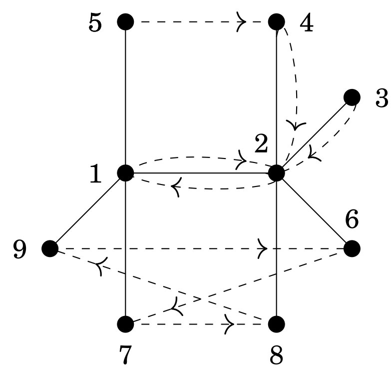

Figure 3.3: The tree X for the second map σ from Example 3.2.

Acknowledgments

The author is deeply thankful to the Armed Forces of Ukraine for keeping Kyiv safe during the preparation of this paper. The author also acknowledges the partial support from Fund (Endowment) of Projects of the MBFVKMA for the development of science, innovations and scientific education at NaUKMA.

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