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Research Article

On the eigenvalues of the distance matrix of graphs with given number of pendant vertices

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Abstract

Let G be a simple connected graph with vertices $v_1, v_2 \dots, v_n$. The distance matrix of G, denoted by $D(G)$, is the $n \times n$ matrix whose $(i,j)^{th}$ element is equal to $d(v_i, v_j)$ (the length of a shortest path between v_i and v_j). Let $\mathbb{P}(n,r)$ be the family of all connected graphs of order n having r pendant vertices. In this paper, we obtain the distance spectrum of various subfamilies of $\mathbb{P}(n,r)$, like pineapple graphs, kite graphs, double star graphs, etc. We also determine the graphs with the largest and smallest spectral radii belonging to these families. Finally, we give a lower bound for the smallest distance eigenvalue of certain kite graphs in terms of minimum transmission.

Keywords: distance matrix; distance spectrum; kite graph; pineapple graph; star graph; distance spectral radius.

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1. Introduction

Let $G = (V, E)$ be a simple graph with the vertex set $V(G)$ and the edge set $E(G)$. The order of G is the number of vertices of G and the size of G is the number of edges of G. Let $d_G(v)$ be the *degree* of v, which is defined as the number of edges incident with v. We also write d_v if the underlying graph is obvious. We use the notation $N(v) = \{u \in V(G) \mid uv \in E(G)\}$ to denote the neighborhood of $v \in V(G)$. By convention, K_n represents the complete graph with order n and $K_{1,n-1}$ represents the star graph with order n. We denote by $K_{a,b}$ the complete bipartite graph with parts V_1 and V_2 such that $|V_1| = a$ and $|V_2| = b$. We refer readers to [12] for other related notations. Given two vertices $u, v \in V(G)$, the *distance* between them is defined as the length of a shortest path between them. We use the notation d_{uv} to represent this distance. The maximum distance between any pair of vertices in G is the *diameter* of G, and is denoted as $d(G)$. The distance matrix of G, denoted as $D(G)$, is the $n \times n$ matrix whose $(i,j)^{th}$ element is equal to $d(v_i, v_j)$. We note that $D(G)$ is real symmetric matrix. Hence all its eigenvalues are real [6]. We denote the eigenvalues of the distance matrix as $\rho_1 \geq \rho_2 \geq \cdots \geq \rho_n$. The distance spectral radius $\rho(G) = \rho_G$ of G is the largest eigenvalue of the distance matrix $D(G)$. The *transmission* $Tr_G(v)$ of a vertex v is defined to be the sum of the distances from v to all the vertices in G, that is,

$$Tr_G(v) = \sum_{u \in V(G)} d_{uv}.$$

For any vertex $v_i \in V(G)$, the transmission $Tr_G(v_i)$ is called the *transmission* degree, shortly denoted by Tr_i or $Tr(v_i)$ and the sequence $\{Tr_1, Tr_2, \dots, Tr_n\}$ is called the transmission degree sequence of the graph. An independent set in a graph is a set of vertices such that no two vertices in the set are adjacent. The size of largest independent set in a graph is called the independence number of the graph. A star independent set is a star induced subgraph. This means all the vertices in this independent set are adjacent to some common vertex outside the independent set, called the central vertex. We use the notation $m_D[a,b]$ to denote the number of eigenvalues of D in the interval $[a,b]$, counted with their multiplicities. The inertia of a matrix M is the triple of integers $(n_+(M), n_0(M), n_-(M))$, where $n_+(M)$, $n_0(M)$ and $n_-(M)$ denote the number of positive, 0 and negative eigenvalues of M, respectively. If $det(M) = 0$, we call M singular; otherwise, we call M non-singular.

Extensive research has been conducted on the spectrum of the distance matrix that was initially investigated in 1971 by Graham and Pollack [9] as part of their study on a data communication problem, see the survey [1]. Our focus will be to investigate the distance spectrum of the connected graphs of order n and having r pendant vertices that belong to the family $\mathbb{P}(n,r)$. We will consider the subfamilies of $\mathbb{P}(n,r)$ which are defined on the next page.

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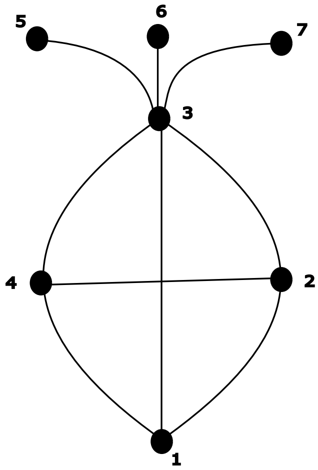

Figure 1.1: Pineapple graph K_4^3.

Pineapple graph K_n^r. The pineapple graph K_n^r is obtained by appending r pendant edges to a single vertex of a complete graph K_{n-r}.

Kite graph $PK_{w,l}$. The kite graph $PK_{w,l}$ is a graph obtained from a clique K_w and a path P_l by adding an edge between a vertex from the clique and an end vertex from the path.

Double star $T(a,b)$. Let S_{a+1} and S_{b+1} be respectively the stars on $a+1$ and $b+1$ vertices with $a,b \geq 1$. Let $T(a,b)$ be the tree obtained by joining the vertex of degree a in S_{a+1} and the vertex of degree b in S_{b+1} by an edge. This tree is called the double star.

The rest of the paper is organized as follows. In Section 2, we obtain the distance spectrum of the kite graph K_n^r. In Section 2, we also determine the graphs having the smallest and the largest spectral radius among the connected graphs of order n, containing a cycle of length $n-r$ and r pendant vertices. We obtain the distance spectrum of the double star $T(a,b)$ in Section 3. In Section 4, we obtain the distance spectrum of the Kite graph $PK_{w,2}$ and find the lower bound for the smallest eigenvalue ρ_n of $PK_{w,2}$ in terms of minimum transmission. We conclude the paper by emphasizing the significance of our findings in Section 5.

2. Distance spectrum of pineapple graphs K_n^r

Bose et al. [4] demonstrated that the pineapple graph K_n^r has the minimum distance spectral radius among all n vertex graphs having r pendant vertices, where $r \neq n-2$. In this section, we obtain the distance spectrum of the pineapple graph K_n^r and find the distribution of its eigenvalues in the interval $[-2,0]$.

Definition 2.1. Let M be an $n \times n$ complex matrix delineated in the block form as given below:

$$M = \begin{pmatrix} M_{11} & M_{12} & \dots & M_{1t} \\ M_{21} & M_{22} & \dots & M_{2t} \\ . & . & . & . \\ . & . & . & . \\ M_{t1} & M_{t2} & \dots & M_{tt} \end{pmatrix}, \tag{2.2}$$

where $n = n_1+n_2+\cdots+n_t$ and M_{ij} is a matrix block with dimension $n_i \times n_j$ for $1 \leq i,j \leq t$. Let b_{ij} be the sum of all elements in M_{ij} divided by the number of rows. In other words, it is the average row sum of M_{ij}, where $1 \leq i,j \leq t$. $B(M) = (b_{ij})$ (or B) is referred to as the quotient matrix of M. If M_{ij} has a constant row sum for all i and j, that is, $M_{ij}e_{ij} = b_{ij}e_{ij}$, then we say B is the equitable quotient matrix, where $e_{ij} = (1,1,\dots,1)^T$. In [20], it was proved that if M is defined as in (2.2) such that $M_{ij} = s_{ij}J_{n_i,n_j}$ for $i \neq j$, and $M_{ii} = s_{ii}J_{n_i,n_i} + p_iI_{n_i}$. Then the equitable quotient matrix of M is $B = (b_{ij})$ with $b_{ij} = s_{ij}n_j$, if $i \neq j$ and $b_{ii} = s_{ii}n_i + p_i$. Moreover,

$$\sigma(M) = \sigma(B) \cup \left\{p_1^{[n_1-1]}, \dots, p_t^{[n_t-1]}\right\}.$$

Let $\mathbb{P}(n,r)$ be the set of all connected graphs of order n and having r pendant vertices. We observe that $\mathbb{P}(1,0) = \{K_1\}$, $\mathbb{P}(3,1) = \mathbb{P}(2,0) = \mathbb{P}(2,1) = \phi$, $P(2,2) = \{K_2\}$ and $P(n,n) = \phi$ for $n \geq 3$. Therefore, we consider $n \geq 3$ and $0 \leq r \leq n-1$. Let K_n^r be the graph obtained by attaching r pendant vertices to any one vertex of K_{n-r}. For example, $K_n^0 = K_n$ and $K_n^{n-1} = K_{1,n}$. Further, we let $\mathcal{K}(n,r)$ be the set of graphs in $\mathbb{P}(n,r)$ such that all r pendant vertices are adjacent to a vertex of degree $n-1$. We note that $K_n^r \in \mathcal{K}(n,r)$. An example of a graph K_n^r, with $n = 7$ and $r = 3$ is shown in Figure 1.1. The following lemmas will be used in the sequel.

Lemma 2.1. [11]. Let G be a connected graph on n vertices. If $S = v_1, v_2, \ldots, v_p$ induces a clique of G with $N(v_i) \setminus S = N(v_j) \setminus S$ for $1 \leq i, j \leq p$. Then -1 is an eigenvalue of $D(G)$ with multiplicity at least $p-1$.

Lemma 2.2. [11]. Let G be a connected graph. If G contains a star independent set of order p, then -2 is an eigenvalue of $D(G)$ with multiplicity at least $p-1$.

The following theorem gives the distance spectrum of K_n^r.

Theorem 2.1. (i) If $n \neq 2, 3$ and $1 \leq r < n-1$, the distance spectrum of K_n^r is $\{-2^{[r-1]}, -1^{[n-r-2]}, x, y, z\}$, where $x,\ y$ and z are the zeros of the polynomial

$$f(x) = x^3 - (n+r-4)x^2 - (2nr + 3n - 2r^2 - 2r - 5)x + (r^2 + 2r + 2 - nr - 2n).$$

(ii) If $r = n-1$, the distance spectrum of K_n^r is $\left\{-2^{[n-2]}, \frac{2n-4 \pm \sqrt{(2n-4)^2 + 4n - 1}}{2}\right\}$.
Moreover, for $n \geq 2$, K_n^r has only one positive distance eigenvalue.

Proof. (i) For $n = 2$, the only possible connected graph is K_2. The distance spectrum of this graph is $\{1, -1\}$. For $r = 3$, the only possible connected graph with at least one pendant vertex is the path P_3, and the distance spectrum of P_3 is $\{-2, 1 \pm \sqrt{3}\}$.
For $n \geq 4$ and $r < n-1$, let $V = \{v_1, v_2, \ldots, v_n\}$ be the vertex set of K_n^r. Assume that $\{v_1, v_2, \ldots, v_{n-r}\}$ is the set of vertices of the induced subgraph K_{n-r} of K_n^r and vertex v_1 is of degree $n-1$. Then, the set $S = \{v_2, v_3, \ldots, v_{n-r}\}$ induces a clique of K_n^r, satisfying $N(v_i) \setminus S = N(v_j) \setminus S$ for $2 \leq i, j \leq n-r$. Therefore, by Lemma 2.1, -1 is an eigenvalue of $D(K_n^r)$ with multiplicity at least $n-r-2$. Similarly, the r pendant vertices of K_n^r form a star independent set and share the same neighborhood, namely vertex v_1. Therefore, by Lemma 2.2, -2 is an eigenvalue of multiplicity at least $r-1$. The remaining three eigenvalues are the eigenvalues of the following equitable quotient matrix

$$B = \begin{pmatrix} 0 & n-r-1 & r \\ 1 & n-r-2 & 2r \\ 1 & 2n-2r-2 & 2r-2 \end{pmatrix},$$

which is obtained from the distance matrix of K_n^r.

By direct computation, we see that the characteristic polynomial of B is

$$f(x) = x^3 - (n+r-4)x^2 - (2nr + 3n - 2r^2 - 2r - 5)x + (r^2 + 2r + 2 - nr - 2n).$$

As seen above, -1 and -2 are the distance eigenvalues of K_n^r with multiplicities $n-r-2$ and $r-1$, respectively. So the possible positive eigenvalues are the zeros of the polynomial $f(x) = x^3 - (n+r-4)x^2 - (2nr + 3n - 2r^2 - 2r - 5)x + (r^2 + 2r + 2 - nr - 2n)$. We can see that there is only one change in signs of the coefficients of $f(x)$, when $n \geq 5$. Hence by Descarte's rule of signs, there is exactly one positive zero of the polynomial $f(x)$. Therefore, K_n^r has exactly one positive eigenvalue. This completes the proof of the first part.
(ii) For $r = n-1$, K_n^r is the star graph S_n. In this case, the $n-1$ pendant vertices of K_n^r form an independent set with the same neighborhood. Therefore, by Lemma 2.2, -2 is an eigenvalue of K_n^r with multiplicity at least $n-2$. The remaining two eigenvalues are eigenvalues of the equitable quotient matrix

$$B' = \begin{pmatrix} 0 & n-1 \\ 1 & 2n-4 \end{pmatrix},$$

where B' is obtained from the distance matrix of K_n^{n-1}.

It is easy to see that the eigenvalues of B' are $\frac{2n-4 \pm \sqrt{(2n-4)^2 + 4n - 1}}{2}$. Hence the complete distance spectrum of K_n^{n-1} is $\left\{-2^{[n-2]}, \frac{2n-4 \pm \sqrt{(2n-4)^2 + 4n - 1}}{2}\right\}$. It can be easily seen that $\frac{2n-4 - \sqrt{(2n-4)^2 + 4n - 1}}{2} < 0$ for all $n \in \mathbb{N}$. Consequently, in this case as well, K_n^r has only one positive eigenvalue.

Hence, from each of the above cases, we see that K_n^r has precisely one positive eigenvalue. □

The following theorem completely describes the distance spectrum of K_n^r for $r < n-1$ and $n \geq 5$.

Theorem 2.2. *Let $\rho_1 \geq \rho_2 \geq \cdots \geq \rho_n$ be the distance spectrum of K_n^r, where $r < n-1$. Then $n+r-3 < \rho_1 < 2n-3$ and $-1 < \rho_2 < 0$. Moreover, for $n \geq 5$, $m_D(K_n^r)[-2,0] = n-2$ or $n-1$.*

Proof. From Theorem 2.1, the distance spectrum of K_n^r is $\{-2^{[r-1]}, -1^{[n-r-2]}, x_1, x_2, x_3\}$, where $x_1 \geq x_2 \geq x_3$ are the zeros of the polynomial

$$f(x) = x^3 - (n+r-4)x^2 - (2nr+3n-2r^2-2r-5)x + (r^2+2r+2-nr-2n).$$

Now, for $0 \leq r < n$, we have $f(0) = r^2+2r+2-nr-2n < 0$ and $f(-1) = -1-n-r+4+2nr+3n-2r^2-2r-5+r^2+2r+2-nr-2n = nr-r^2-r \geq 0$.

Using the intermediate value theorem, it follows that at least one zero of $f(x)$, say x_2, lies in the interval $[-1,0)$. Since $x_1+x_2+x_3 = n+r-4$ and $-1 \leq x_2 < 0$, therefore $n+r-4 < x_1+x_3 < n+r-3$. Also the maximum row sum of the matrix B is $2n-3$. So $x_1 < 2n-3$. This implies that $x_3 > -n+r-1$.

From Theorem 2.1, we see that -2 and -1 are the distance eigenvalues of K_n^r with multiplicities $r-1$ and $n-r-2$, respectively. Also, as seen above, there is at least one eigenvalue of K_n^r in the interval $[-1,0)$. As K_n^r has only one positive eigenvalue, so $[-2,0]$ contains either $n-2$ or $n-1$ eigenvalues of K_n^r. $\square$

Remark 2.1. *We have established that the distance matrix of K_n^r has one positive eigenvalue and $n-1$ negative eigenvalues. Consequently, the inertia of $D(K_n^r)$ is $(1,0,n-1)$. This indicates that the distance matrix of K_n^r is non-singular, with a positive determinant when n is odd and a negative determinant when n is even.*

Observation 1. Suppose the connected graph G has an edge $e = uv$ satisfying $G' = G-e$ being connected. The distance matrix of $G-e$ is denoted by D'. Deleting the edge e will not generate shorter paths than those in G. Thus, $d_{ij} \leq d'_{ij}$ for all $i,j \in V$. We have $1 = d_{uv} < d'_{uv}$. In view of the Perron-Frobenius theorem, $\rho(G) < \rho(G-e)$. Furthermore, for any spanning tree T of G, $\rho(G) \leq \rho(T)$ holds.

With this observation and the following lemma, we find the graph with the maximum distance spectral radius among all graphs having r pendant vertices and containing a cycle of length $n-r$.

Lemma 2.3. [20] Let G be a graph with a clique K_s of order $(s \geq 2)$ and u and v be two vertices on the clique with p and q pendant vertices respectively, where $deg(v) = q+s-1$ in G. If $G' = G - vw + uw$, where w is a pendant vertex adjacent to v in G, then $\rho(G) > \rho(G')$ for $p \geq q \geq 1$.

Let G be a connected graph of order n with r pendant vertices. Let G contain at least one cycle, say C, of length $n-r$. Clearly, this cycle C of length $n-r$ is the cycle of the largest length, since G contains r pendant vertices. Let $E(H)$ be the set of pendant edges of G. From G, we delete the edges of $E(G-H-C)$, that is, we delete those non-pendant edges of G which are not in C. The resulting graph, denoted by C_n^r, consists of the cycle of length $n-r$ and r pendant edges. In Figure 2.1, we have obtained C_9^3 of the first graph in two ways. Note that the pendant vertices are adjacent on the same vertices as these are in the original graph and C_9^3 consists of the cycle C_6 with three pendant edges as these are in the original graph.

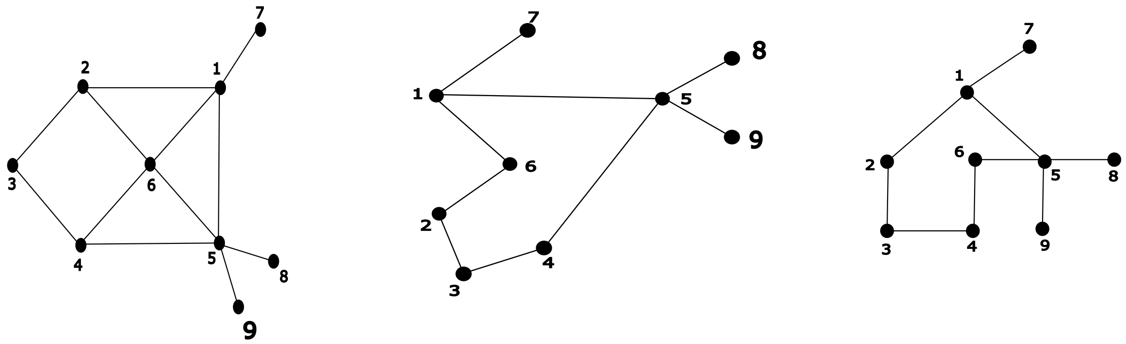

Figure 2.1: The C_9^3 graphs of the graph G.

Now, we show that the spectral radius of G lies between the spectral radii of C_n^r and K_n^r.

Theorem 2.3. Let G be a connected graph of order n with r pendant vertices. If G has a cycle of length $n-r$, then $\rho(C_n^r) > \rho(G) > \rho(K_n^r)$.

Proof. From G, delete all the edges that do not shorten the length of the largest cycle and keep G connected. Obviously, after the deletion of all such edges, the resulting graph is C_n^r. Then, from Observation 1, we have $\rho(C_n^r) > \rho(G)$.

For the other part, we add all possible edges between the non-adjacent vertices of the cycle so that the cycle becomes a clique K_{n-r}. Next, we repeatedly perform the transformation of deleting the pendant edges and adding them on some fixed vertex of the clique, we just formed, so that the final graph is K_n^r. Now, using Lemma 2.3 and Observation 1, we get $\rho(G) > \rho(K_n^r)$. This completes the proof. □

3. Distance spectrum of $\mathbb{T}(a, b)$

The double star graph $2S_n$ is a graph formed by duplicating the star graph S_n and connecting two vertices of degree n with an edge. Collins [5] derived the characteristic polynomial of $2S_n$. Extending this concept, we define the double star graph $T(a, b)$ as the union of two arbitrary star graphs S_a and S_b, with their central vertices joined by an edge. We may note that $2S_n$ can be obtained from $T(a, b)$ by simply putting $a = b = n$. Let T_n be the family of double stars with $a + b = n - 2$ and $a, b \geq 1$, that is, $T_n = \{T(a, b) : a + b = n - 2, a, b \geq 1\}$. The following theorem gives the distance spectrum of $T(a, b)$.

Theorem 3.1. *For the tree $T(a, b)$ with $a + b = n - 2$, the distance spectrum is $\{-2^{n-4}, x_1, x_2, x_3$ and $x_4\}$, where x_1, x_2, x_3 and x_4 are the zeros of the polynomial*

$$P(x) = x^4 - 2(a + b - 2)x^3 - (5ab + 9a + 9b - 3)x^2 - 4(ab + 3a + 3b + 1)x - 4(a + b + 1).$$

For $n \geq 3$, $T(a, b)$ has only one positive distance eigenvalue.

Proof. Let $V(S_{a+1}) = \{u, u_1, u_2, \ldots, u_a\}$ and $V(S_{b+1}) = \{v, v_1, \ldots, v_b\}$ be respectively the vertex set of the stars S_{a+1} and S_{b+1}. Assume the vertices u and v are central vertices of the stars S_{a+1} and S_{b+1}, respectively . Then the vertex set of $T(a, b)$ is $V(T(a, b)) = V(S_{a+1}) \cup V(S_{b+1}) = \{v, v_1, \ldots, v_b, u, u_1, \ldots, u_a\}$. By appropriately labelling the vertices of $T(a, b)$, it is possible to write the distance matrix of $T(a, b)$ in the form

$$D(T(a,b)) = \begin{pmatrix} 0 & J_{1\times a} & J_{1\times 1} & 2J_{1\times b} \\ J_{a\times 1} & (-2I+2J)_{a\times a} & 2J_{a\times 1} & 3J_{a\times b} \\ J_{1\times 1} & 2J_{1\times a} & 0 & J_{1\times b} \\ 2J_{b\times 1} & 3J_{b\times a} & J_{b\times 1} & (-2I+2J)_{b\times b} \end{pmatrix}.$$

The set $S = \{u_1, u_2, \ldots, u_a\}$ forms a star independent set with a common neighborhood outside S. According to Lemma 2.2, this ensures that -2 is an eigenvalue with a multiplicity of at least $a - 1$. Similarly, the set $S' = \{v_1, v_2, \ldots, v_b\}$ is an independent set with a shared neighborhood outside S', so again by Lemma 2.2, -2 is an eigenvalue with a multiplicity of at least $b - 1$. Thus, the total multiplicity of -2 is at least $a + b - 2 = n - 4$. The remaining eigenvalues are the eigenvalues of the following equitable quotient matrix B obtained from $D(T(a, b))$

$$B = \begin{pmatrix} 0 & a & 1 & 2b \\ 1 & 2a-2 & 2 & 3b \\ 1 & 2a & 0 & b \\ 2 & 3a & 1 & 2b-2 \end{pmatrix}.$$

It can be seen by simple calculations that the characteristic polynomial of B is

$$P(x) = x^4 - 2(a + b - 2)x^3 - (5ab + +9a + 9b - 3)x^2 - 4(ab + 3a + 3b + 1)x - 4(a + b + 1).$$

From the above discussion, we note that -2 is the distance eigenvalue of $T(a, b)$ with multiplicity $n - 4$. So the possible positive eigenvalues are the zeros of the polynomial $P(x) = x^4 - 2(a + b - 2)x^3 - (5ab + +9a + 9b - 3)x^2 - 4(ab + 3a + 3b + 1)x - 4(a + b + 1)$. By Descarte's rule of signs, there is only one change in the signs for the coefficients of $P(x)$ when $n \geq 3$. Thus, there is exactly one positive zero of the polynomial $P(x)$. Therefore, $T(a, b)$ has exactly one positive eigenvalue. Note that this positive eigenvalue is in fact the distance spectral radius of $T(a, b)$. □

The spectrum of $T(a, b)$ is completely characterized by the following theorem.

Theorem 3.2. *Let $\rho_1 \geq \rho_2 \geq \cdots \geq \rho_n$ be the distance spectrum of $T(a, b)$. Then, for $a > b > 2$ we have $2b + 2a - 0.5 < \rho_1 < 3a + 2b + 1$, $-0.5 < \rho_2 < 0$, $-1 < \rho_3 < -0.5$, $\rho_i = -2$ for $i = 4, 5, \ldots, n - 1$ and $-3a < \rho_n < -3$.*

Proof. From Theorem 3.1, clearly -2 is an eigenvalue of $D(T(a, b)$ of multiplicity $n - 4$ and the remaining four eigenvalues are the zeros of the polynomial

$$P(x) = x^4 - 2(a + b - 2)x^3 - (5ab + 9a + 9b - 3)x^2 - 4(ab + 3a + 3b + 1)x - 4(a + b + 1).$$

Now, we use the intermediate value property to locate these four zeros of $P(x)$. We have

$$\begin{aligned}
P(0) &= -4(a+b+1) < 0,\\
P(-0.5) &= 0.0625 + 0.25(a+b-2) - 0.25(5ab+9a+9b-3) + 2ab + 6a + 6b + 2 - 4(a+b+1)\\
&= -0.1875a + 0.75ab - 1.6875 > 0,\ for\ ab \geq 4,\\
P(-1) &= 1 + 2a + 2b - 4 - 5ab - 9a - 9b + 3 + 4ab + 12a + 12b + 4 - 4a - 4b - 4\\
&= a + b - ab < 0,\\
P(-3) &= 81 + 54a + 54b - 108 - 45ab - 81a - 81b + 9 + 12ab + 36a + 36b + 12 - 4a - 4b - 4\\
&= -10 + 5a + 5b - 33ab < 0,\\
P(-3a) &= 135a^4 + 9a^3b - 189a^3 - 69a^2b + 63a^2 + 36ab + 8a - 4b - 4 > 0.
\end{aligned}$$

Using the intermediate value theorem, we conclude that there exists at least one zero of the polynomial $P(x)$ in each of the intervals $(-0.5, 0), (-1, -0.5)$ and $(-3a, -3)$. As the polynomial $p(x)$ has exactly three negative zeros, so there exists exactly one zero of $P(x)$ in each of these intervals. Therefore, if $x_1 \leq x_2 \leq x_3 \leq x_4$ are the zeros of $P(x)$, then $-0.5 < x_1 < 0, -1 < x_2 < -0.5$ and $-3a < x_3 < -3$. Since the sum of these zeros is $2(a+b-2)$, so $2a + 2b - 0.5 < x_4 < 3a + 2b + 1$. Using the fact that the zeros of the characteristic polynomial are the eigenvalues of the corresponding matrix, we have $2b + 2a - 0.5 < \rho_1 < 3a + 2b + 1, -0.5 < \rho_2 < 0, -1 < \rho_3 < -0.5, \rho_i = -2$ for $i = 4, 5, \ldots, n-1$ and $-3a < \rho_n < -3$. □

Remark 3.1. *We have demonstrated that the distance matrix of $T(a, b)$ has exactly one positive eigenvalue and $n-1$ negative eigenvalues. Therefore, the inertia of $D(T(a, b))$ is $(1, 0, n-1)$. This implies that the distance matrix of $T(a, b)$ is non-singular, with a positive determinant when n is odd and a negative determinant when n is even.*

4. Distance spectrum of the Kite graph $PK_{w,2}$

Definition 4.1. For integers w, l, n with $w + l = n$, let $PK_{w,l}$ be the graph obtained from the complete graph K_w and the path P_l by adding an edge between any vertex of K_w and a pendant vertex of P_l. We call $PK_{w,l}$ as the $Kite\ graph$. [7, 19]. We note that $PK_{w.l} \in \mathbb{P}(n, r)$ and $K_n^1 = PK_{n-1,1}$. As an example, the graph $PK_{5,3}$ is shown in Figure 4.1.

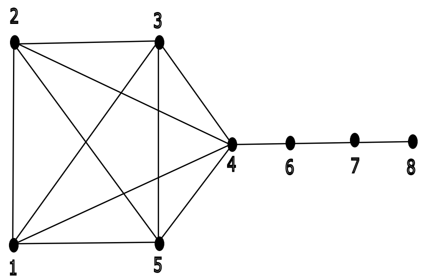

Figure 4.1: Kite graph $PK_{5,3}$.

When $l = 1$, $PK_{w,l}$ represents the Pineapple graph K_n^1. The distance spectrum of this graph has been extensively discussed in Theorem 2.1. Now, we examine the distance spectrum of $PK_{w,l}$ for $l = 2$. The subsequent lemma provides the distance spectrum of $PK_{w,2}$.

Lemma 4.1. *For $w+2 = n$, the distance spectrum of $PK_{w,2}$ is $\{-1^{[n-4]}, a, b, c, d\}$, where a, b, c, d are the zeros of the polynomial*

$$f(x) = x^4 - (w-2)x^3 - (14w-8)x^2 - (22w-12)x - (8w-4).$$

Moreover, for $w \geq 2$, $D(PK_{w,2})$ has only one positive distance eigenvalue.

Proof. Let $V = \{v_1, v_2, \ldots, v_w\}$ be the vertex set of the complete graph K_w and $U = \{u_1, u_2\}$, where $w+2 = n$, be the vertex set of the path P_2. Suppose that an edge $v_w u_1$ is added between the pendant vertex u_1 of P_2 and the vertex v_w of K_w, so that the resulting graph is $PK_{w,2}$. We label the rows and columns of $D(PK_{w,2})$ in the order $v_1, \ldots, v_w, u_1$ and u_2, so that the distance matrix of $PK_{w,2}$ has the form

$$D(PK_{w,2}) = \begin{pmatrix} 0 & J_{1\times w-1} & J_{1\times 1} & 2J_{1\times 1} \\ J_{w-1\times 1} & (-I+J)_{w-1\times w-1} & 2J_{w-1\times 1} & 3J_{w-1\times 1} \\ J_{1\times 1} & 2J_{1\times w-1} & (-I+J)_{1\times 1} & J_{1\times 1} \\ 2J_{1\times 1} & 3J_{1\times w-1} & J_{1\times 1} & (-I+J)_{1\times 1} \end{pmatrix}.$$

The set of vertices $T = \{v_1, v_2, \ldots, v_{w-1}\}$ induces a clique of $PK_{w,2}$ and the vertices of T share the same neighborhood outside T. From Lemma 2.1, it follows that -1 is an eigenvalue of multiplicity at least $n-4$. The remaining eigenvalues are the eigenvalues of the matrix B, where

$$B = \begin{pmatrix} 0 & w-1 & 1 & 2 \\ 1 & w-2 & 2 & 3 \\ 1 & 2w-2 & 0 & 1 \\ 2 & 3w-3 & 1 & 0 \end{pmatrix}.$$

The characteristic polynomial of this matrix is the determinant of $B - xI$,

$$\begin{vmatrix} -x & w-1 & 1 & 2 \\ 1 & (w-2)-x & 2 & 3 \\ 1 & 2w-2 & -x & 1 \\ 2 & 3w-3 & 1 & -x \end{vmatrix} = x^4 - (w-2)x^3 - (14w-8)x^2 - (22w-12)x - (8w-4).$$

We now prove the second part of the lemma. Since -1 is an eigenvalue of $D(PK_{w,2})$ of multiplicity $n-4$, so the possible positive eigenvalues are the zeros of the polynomial

$$f(x) = x^4 - (w-2)x^3 - (14w-8)x^2 - (22w-12)x - (8w-4).$$

For $w \geq 2$, we observe that there is only one change in the signs of $f(x)$. Therefore, by Descarte's rule of sign, $f(x)$ has exactly one positive zero. This implies that $D(PK_{w,2})$ has exactly one positive eigenvalue, for $w \geq 2$. □

The following theorem gives the lower bound for the smallest distance eigenvalue of $D(PK_{w,2})$ in terms of minimum transmission.

Theorem 4.1. *Let w be an integer with $w \geq 4$ and $n = w+2$. The smallest distance eigenvalue of $PK_{w,2}$ satisfies $\rho_n > -Tr_{min}$, where Tr_{min} is the minimum transmission of $PK_{w,2}$.*

Proof. From Lemma 4.1, for $w+2 = n$, the distance spectrum of $PK_{w,2}$ is $\{-1^{[n-4]}, a, b, c, d\}$, where a, b, c, d are the zeros of the polynomial

$$f(x) = x^4 - (w-2)x^3 - (14w-8)x^2 - (22w-12)x - (8w-4).$$

To find a lower bound for $\rho_n(PK_{w,2})$, we first locate the three negative zeros of $f(x)$. We have

$$\begin{aligned} f(0) &= -(8w-4) < 0, \ \ for\ w \geq 2, \\ f(-0.7) &= 0.2401 + 0.343w - 0.646 - 6.86w + 3.92 + 15.4w - 8.4 - 8w + 4 \\ &= 0.883w - 0.9259 > 0, \ \ for\ w \geq 2, \\ f(-1) &= 1 + w - 2 - 14w + 8 + 22w - 12 - 8w + 4 \\ &= w - 1 > 0, \ \ for\ w \geq 2, \\ f(-2) &= 16 + 8w - 16 - 56w + 32 + 44w - 24 - 8w + 4 \\ &= -12w + 12 < 0, \ \ for\ w \geq 3, \\ f(-n) &= n^4 + (w-2)n^3 - (14w-8)n^2 + (22w-12)n - 8w + 4 \\ &= n^4 + (n-4)n^3 - 14(n-2)n^2 + 8n^2 + 22(n-2)n - 12n - 8(n-2) + 4 \\ &= 2n^4 - 18n^3 + 58n^2 - 64n + 20 > 0, \ \ for\ n \geq 6. \end{aligned}$$

By using the intermediate value property, we see that there exists exactly one zero of $f(x)$ in each of the intervals $(-0.7, 0), (-2, -1)$ and $(-n, -2)$. Therefore, $-0.7 < \rho_2 < 0, -2 < \rho_{n-1} < -1$ and $-n < \rho_n < -2$. Also, the transmission of the vertices of $PK_{w,2}$ is n or $n+2$ or $2n-4$ or $3n-6$, the vertex of K_w adjacent to the pendant vertex of P_2 has the minimum transmission n. Thus, we have $-Tr_{min} = -n < \rho_n(PK_{w,2}) < -2$. □

Remark 4.1. *We have determined that the distance matrix of $PK_{w,2}$ has precisely one positive eigenvalue, with the remaining $n-1$ eigenvalues being negative. Therefore, the inertia of $D(PK_{w,2})$ is $(1, 0, n-1)$. This reveals that the distance matrix of $PK_{w,2}$ is non-singular, exhibiting a positive determinant when n is odd and a negative determinant when n is even.*

5. Conclusion

We have computed the distance spectrum of various graph families with r pendant vertices. The upper and lower bounds for the smallest and the largest eigenvalues of these families have also been obtained. It is generally recognized that locating the extremal graphs that correspond to these bounds is a challenging task. Therefore, it would be interesting to explore and examine the issue of identifying the extremal cases of the bounds presented in this paper. Further it will be interesting to investigate distance matrix on the basis of some recent work on spectra of distance Laplacian matrix [1–3, 8, 10, 13–18].

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