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Research Article

The general first Zagreb index conditions for Hamiltonian and traceable graphs

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Abstract

The general first Zagreb index of a graph G is defined as the sum of the αth powers of the vertex degrees of G, where α is a real number such that $\alpha \neq 0$ and $\alpha \neq 1$. In this paper, sufficient conditions involving the general first Zagreb index, with $\alpha > 1$ or $\alpha < 0$, are presented for the Hamiltonian and traceable graphs.

Keywords: general first Zagreb index; Hamiltonian graph; traceable graph.

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1. Introduction

Only finite undirected graphs without loops and multiple edges are considered in this paper. Notation and terminology not defined here follow those in [2]. Let $G = (V(G), E(G))$ be a graph with n vertices and e edges. The degree of a vertex v is denoted by $d_G(v)$. The symbols δ and Δ are used to denote the minimum degree and maximum degree of G, respectively. A set of vertices in a graph G is independent if the vertices in that set are pairwise nonadjacent. A maximum independent set in a graph G is an independent set of the largest possible size. The independence number, denoted $\beta(G)$, of a graph G is the cardinality of a maximum independent set in G. For disjoint vertex subsets X and Y of $V(G)$, the notation $E_G(X, Y)$ is used to denote the set of all those edges of G whose one end vertex is in X and the other end vertex is in Y; namely, $E_G(X, Y) := \{e : e = xy \in E(G), x \in X, y \in Y\}$. A cycle C in a graph G is called a Hamiltonian cycle of G if C contains all the vertices of G. A graph G is called Hamiltonian if G has a Hamiltonian cycle. A path P in a graph G is called a Hamiltonian path of G if P contains all the vertices of G. A graph G is called traceable if G has a Hamiltonian path.

The first Zagreb index was introduced by Gutman and Trinajstić in [5]. For a graph G, its first Zagreb index is defined as $\sum_{v \in V(G)} d_G^2(v)$. Li and Zheng in [10] extended the definition of the first Zagreb index by introducing the concept of the general first Zagreb index. The general first Zagreb index, denoted $M_\alpha(G)$, of a graph G is defined as

$$M_\alpha(G) = \sum_{v \in V(G)} d_G^\alpha(v),$$

where α is a real number such that $\alpha \neq 0$ and $\alpha \neq 1$. Some results on the general first Zagreb index of a graph can be found in the survey paper [1]. In recent years, several sufficient conditions based on the first Zagreb index or the general first Zagreb index for the Hamiltonian properties of graphs have been obtained. Some of them can be found in [6–9, 11]. In the present paper, using the general first Zagreb index of a graph with $\alpha > 1$ or $\alpha < 0$, sufficient conditions for the Hamiltonian and traceable graphs are presented. In what follows, the main results of this paper are stated.

Theorem 1.1. *Let G be a k-connected ($k \geq 2$) graph with $n \geq 3$ vertices and e edges. Assume α is a real number.*

(i). *If $\alpha > 1$ and*

$$M_\alpha \geq e^{\frac{\alpha}{2\alpha - 1}} \left((k+1)\Delta^{2\alpha}\right)^{\frac{\alpha-1}{2\alpha-1}} + (n - k - 1)\Delta^\alpha,$$

then G is Hamiltonian, where Δ is the maximum degree of G.

(ii). *If $\alpha < 0$ and*

$$M_\alpha \geq e^{\frac{\alpha}{2\alpha - 1}} \left((k+1)\delta^{2\alpha}\right)^{\frac{\alpha-1}{2\alpha-1}} + (n - k - 1)\delta^\alpha,$$

then G is Hamiltonian, where δ is the minimum degree of G.

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Theorem 1.2. *Let G be a k-connected ($k \geq 1$) with $n \geq 9$ vertices and e edges. Assume α is a real number.*

(i). *If $\alpha > 1$ and*

$$M_\alpha \geq e^{\frac{\alpha}{2\alpha-1}} \left((k+2)\Delta^{2\alpha}\right)^{\frac{\alpha-1}{2\alpha-1}} + (n-k-2)\Delta^\alpha,$$

then G is traceable, where Δ is the maximum degree of G.

(ii). *If $\alpha < 0$ and*

$$M_\alpha \geq e^{\frac{\alpha}{2\alpha-1}} \left((k+2)\delta^{2\alpha}\right)^{\frac{\alpha-1}{2\alpha-1}} + (n-k-2)\delta^\alpha,$$

then G is traceable, where δ is the minimum degree of G.

2. Lemmas

This section provides the known results that are used in the proofs of the main results of this paper.

Lemma 2.1 (see [4]). *Let G be a k-connected graph with order $n \geq 3$ and independence number β. If $\beta \leq k$, then G is Hamiltonian.*

Lemma 2.2 (see [4]). *Let G be a k-connected graph with order n and independence number β. If $\beta \leq k+1$, then G is traceable.*

Lemma 2.3 (see [12]). *Let G be a balanced bipartite graph of order $2n$ with bipartition (A, B). If $d(x) + d(y) \geq n+1$ for any $x \in A$ and any $y \in B$ with $xy \notin E(G)$, then G is Hamiltonian.*

The next lemma follows from the proof of Proposition 1 on Page 145 in [3].

Lemma 2.4 (see [3]). *Let a_1, a_2, ..., a_t be positive real numbers. Suppose that p is a real number with $p < 0$ or $p > 1$. Then*

$$\sum_{i=1}^{t} a_i^p \leq \left(\sum_{i=1}^{t} a_i\right)^{\frac{p}{2p-1}} \left(\sum_{i=1}^{t} a_i^{2p}\right)^{\frac{p-1}{2p-1}}.$$

3. Proofs of Theorems 1.1 and 1.2

Proof of Theorem 1.1. Let G be a k-connected ($k \geq 2$) graph with $n \geq 3$ vertices and e edges. Assume that α is a real number. Suppose that G is not Hamiltonian. Then Lemma 2.1 implies that $\beta \geq k+1$. Also, we have that $n \geq 2\delta+1 \geq 2k+1$ otherwise $\delta \geq k \geq n/2$ and G is Hamiltonian. Let $I_1 := \{ u_1, u_2, ..., u_\beta \}$ be a maximum independent set in G. Then $I := \{ u_1, u_2, ..., u_{k+1} \}$ is an independent set in G. Thus,

$$\sum_{u \in I} d(u) = |E_G(I, V - I)| \leq \sum_{v \in V - I} d(v).$$

Since

$$\sum_{u \in I} d(u) + \sum_{v \in V-I} d(v) = 2e,$$

we have that

$$\sum_{u \in I} d(u) \leq e \leq \sum_{v \in V-I} d(v).$$

(i). Assume that $\alpha > 1$. Applying Lemma 2.4 with $t = k+1$, $p = \alpha$, and $a_i = d(u_i)$, where $1 \leq i \leq k+1$, we have that

$$\sum_{i=1}^{k+1} d^\alpha(u_i) \leq \left(\sum_{i=1}^{k+1} d(u_i)\right)^{\frac{\alpha}{2\alpha-1}} \left(\sum_{i=1}^{k+1} d^{2\alpha}(u_i)\right)^{\frac{\alpha-1}{2\alpha-1}}$$

$$\leq e^{\frac{\alpha}{2\alpha-1}} \left((k+1)\Delta^{2\alpha}\right)^{\frac{\alpha-1}{2\alpha-1}}.$$

Thus,

$$\begin{aligned}
e^{\frac{\alpha}{2\alpha-1}}\left((k+1)\Delta^{2\alpha}\right)^{\frac{\alpha-1}{2\alpha-1}}+(n-k-1)\Delta^{\alpha}\leq M_\alpha &= \sum_{u\in I} d^\alpha(u)+\sum_{v\in V-I} d^\alpha(v)\\
&\leq \left(\sum_{i=1}^{k+1} d(u_i)\right)^{\frac{\alpha}{2\alpha-1}}\left(\sum_{i=1}^{k+1} d^{2\alpha}(u_i)\right)^{\frac{\alpha-1}{2\alpha-1}}+\sum_{v\in V-I} d^\alpha(v)\\
&\leq e^{\frac{\alpha}{2\alpha-1}}\left((k+1)\Delta^{2\alpha}\right)^{\frac{\alpha-1}{2\alpha-1}}+(n-k-1)\Delta^{\alpha}.
\end{aligned}$$

Therefore,

$$\begin{aligned}
M_\alpha &= \sum_{u\in I} d^\alpha(u)+\sum_{v\in V-I} d^\alpha(v)\\
&= \left(\sum_{i=1}^{k+1} d(u_i)\right)^{\frac{\alpha}{2\alpha-1}}\left(\sum_{i=1}^{k+1} d^{2\alpha}(u_i)\right)^{\frac{\alpha-1}{2\alpha-1}}+\sum_{v\in V-I} d^\alpha(v)\\
&= e^{\frac{\alpha}{2\alpha-1}}\left((k+1)\Delta^{2\alpha}\right)^{\frac{\alpha-1}{2\alpha-1}}+(n-k-1)\Delta^{\alpha}.
\end{aligned}$$

Hence,

$$\begin{aligned}
\sum_{u\in I} d^\alpha(u) &= \left(\sum_{i=1}^{k+1} d(u_i)\right)^{\frac{\alpha}{2\alpha-1}}\left(\sum_{i=1}^{k+1} d^{2\alpha}(u_i)\right)^{\frac{\alpha-1}{2\alpha-1}}\\
&= e^{\frac{\alpha}{2\alpha-1}}\left((k+1)\Delta^{2\alpha}\right)^{\frac{\alpha-1}{2\alpha-1}}
\end{aligned}$$

and

$$\sum_{v\in V-I} d^\alpha(v) = (n-k-1)\Delta^{\alpha}.$$

Consequently, we have

$$\sum_{u\in I} d(u)=e,\ d(u)=\Delta,\ d(v)=\Delta,$$

where $u\in I$ and $v\in V-I$. Thus, $V-I$ is independent. Since

$$\Delta|I|=|E_G(I,V-I)|=\Delta|V-I|$$

and

$$|I|=|V-I|=k+1,$$

by Lemma 2.3, we have that G is Hamiltonian, a contradiction. This completes the proof of Theorem 1.1(i).

(ii) Now, we assume that $\alpha<0$. Applying Lemma 2.4 with $t=k+1$, $p=\alpha$, and $a_i=d(u_i)$, where $1\leq i\leq k+1$, we have that

$$\begin{aligned}
\sum_{i=1}^{k+1} d^\alpha(u_i) &\leq \left(\sum_{i=1}^{k+1} d(u_i)\right)^{\frac{\alpha}{2\alpha-1}}\left(\sum_{i=1}^{k+1} d^{2\alpha}(u_i)\right)^{\frac{\alpha-1}{2\alpha-1}}\\
&\leq e^{\frac{\alpha}{2\alpha-1}}\left((k+1)\delta^{2\alpha}\right)^{\frac{\alpha-1}{2\alpha-1}}.
\end{aligned}$$

Thus,

$$\begin{aligned}
e^{\frac{\alpha}{2\alpha-1}}\left((k+1)\delta^{2\alpha}\right)^{\frac{\alpha-1}{2\alpha-1}}+(n-k-1)\delta^{\alpha}\leq M_\alpha &= \sum_{u\in I} d^\alpha(u)+\sum_{v\in V-I} d^\alpha(v)\\
&\leq \left(\sum_{i=1}^{k+1} d(u_i)\right)^{\frac{\alpha}{2\alpha-1}}\left(\sum_{i=1}^{k+1} d^{2\alpha}(u_i)\right)^{\frac{\alpha-1}{2\alpha-1}}+\sum_{v\in V-I} d^\alpha(v)\\
&\leq e^{\frac{\alpha}{2\alpha-1}}\left((k+1)\delta^{2\alpha}\right)^{\frac{\alpha-1}{2\alpha-1}}+(n-k-1)\delta^{\alpha}.
\end{aligned}$$

Therefore,

$$
\begin{aligned}
M_\alpha &= \sum_{u\in I} d^\alpha(u) + \sum_{v\in V-I} d^\alpha(v)\\
&= \left(\sum_{i=1}^{k+1} d(u_i)\right)^{\frac{\alpha}{2\alpha-1}} \left(\sum_{i=1}^{k+1} d^{2\alpha}(u_i)\right)^{\frac{\alpha-1}{2\alpha-1}} + \sum_{v\in V-I} d^\alpha(v)\\
&= e^{\frac{\alpha}{2\alpha-1}} \left((k+1)\delta^{2\alpha}\right)^{\frac{\alpha-1}{2\alpha-1}} + (n-k-1)\delta^\alpha.
\end{aligned}
$$

Hence,

$$
\begin{aligned}
\sum_{v\in I} d^\alpha(v) &= \left(\sum_{i=1}^{k+1} d(u_i)\right)^{\frac{\alpha}{2\alpha-1}} \left(\sum_{i=1}^{k+1} d^{2\alpha}(u_i)\right)^{\frac{\alpha-1}{2\alpha-1}}\\
&= e^{\frac{\alpha}{2\alpha-1}} \left((k+1)\delta^{2\alpha}\right)^{\frac{\alpha-1}{2\alpha-1}}
\end{aligned}
$$

and

$$\sum_{v\in V-I} d^\alpha(v) = (n-k-1)\delta^\alpha.$$

So,

$$\sum_{u\in I} d(u) = e,\ d(u) = \delta,\ d(v) = \delta,$$

where $u \in I$ and $v \in V - I$. Thus, $V - I$ is independent. Since $\delta|I| = |E_G(I, V-I)| = \delta|V-I|$ and $|I| = |V-I| = k+1$. By Lemma 2.3, we have that G is Hamiltonian, a contradiction. This completes the proof of Theorem 1.1(ii). □

Proof of Theorem 1.2. Let G be a k-connected ($k \geq 1$) graph with $n \geq 9$ vertices and e edges. Assume that α is a real number. Suppose that G is not traceable. Then Lemma 2.2 implies that $\beta \geq k+2$. Also, we have that $n \geq 2\delta + 2 \geq 2k+2$ otherwise $\delta \geq k \geq (n-1)/2$ and G is traceable. Let $I_1 := \{u_1, u_2, ..., u_\beta\}$ be a maximum independent set in G. Then $I := \{u_1, u_2, ..., u_{k+2}\}$ is an independent set in G. Thus, we have

$$\sum_{u\in I} d(u) = |E_G(I, V-I)| \leq \sum_{v\in V-I} d(v).$$

Since

$$\sum_{u\in I} d(u) + \sum_{v\in V-I} d(v) = 2e,$$

we have that

$$\sum_{u\in I} d(u) \leq e \leq \sum_{v\in V-I} d(v).$$

Since the proofs of Theorem 1.2(i) and Theorem 1.2(ii) are similar to the proofs of Theorem 1.1(i) and Theorem 1.1(ii), respectively, the remaining proof of Theorem 1.2 is skipped here. □

From the proof of Theorem 1.1, the next result follows.

Corollary 3.1. *Let G be a graph with n vertices and $e \geq 1$ edges. Assume that α is a real number.*

(i). *If $\alpha > 1$, then*

$$M_\alpha \leq e^{\frac{\alpha}{2\alpha-1}} \left(\beta\Delta^{2\alpha}\right)^{\frac{\alpha-1}{2\alpha-1}} + (n-\beta)\Delta^\alpha$$

with equality if and only if G is $K_{\frac{n}{2},\frac{n}{2}}$.

(ii). *If $\alpha < 0$, then*

$$M_\alpha \leq e^{\frac{\alpha}{2\alpha-1}} \left(\beta\delta^{2\alpha}\right)^{\frac{\alpha-1}{2\alpha-1}} + (n-\beta)\delta^\alpha$$

with equality if and only if G is $K_{\frac{n}{2},\frac{n}{2}}$.

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