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Research Article

Tropical geometry of Rado matroids

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Abstract

In this article, we characterize the products of simplicial generators for the Chow ring of a loopless matroid, extending a result of Backman, Eur, and Simpson [*J. Eur. Math. Soc.*, DOI: 10.4171/JEMS/1350]. We prove that the stable intersection of a collection of tropical hyperplanes centered at the origin with the Bergman fan of a matroid is the Bergman fan of the dual of a certain Rado matroid.

Keywords: tropical geometry; Chow ring; Rado matroid.

2020 Mathematics Subject Classification: 05B35, 05B40, 14T05, 14C17.

1. Introduction

The Chow ring $A^\bullet(M)$ of a loopless matroid M on ground set E was introduced by Feichtner and Yuzvinsky in [5] as a generalization of the cohomology ring of De Concini and Procesi's wonderful compactification of the complement of a hyperplane arrangement [3]. The importance of the Chow ring was demonstrated by Adiprasito, Huh, and Katz in the proof of the Heron-Rota-Welsh conjecture [1]. Feichtner and Yuzvinsky define $A^\bullet(M)$ to be the graded ring $\mathbb{R}[z_F \mid F \in \mathscr{L}_M - \varnothing]$ modulo the ideals $\langle z_F z_{F'} \mid F, F' \text{ incomparable}\rangle$ and $\langle \sum_{F \supseteq a} z_F \mid a \in \mathfrak{A}_M\rangle$, where $\mathscr{L}_M$ denotes the lattice of flats of M and $\mathfrak{A}_M$ the set of atoms in $\mathscr{L}_M$.

In [2], Backman, Eur, and Simpson introduced a set of generators for $A^\bullet(M)$ called the *simplicial generators* (independently defined by Yuzvinsky [13] in the context of linear subspace arrangement complements). The simplicial generators are defined, for each nonempty flat F of M, by $h_F(M) = -\sum_{G \supseteq F} z_G(M) \in A^1(M)$. We write h_F for $h_F(M)$ when M is clear from context. We denote by $A^\bullet_\nabla(M)$ the presentation of the Chow ring of M by the simplicial generators:

$$A^\bullet_\nabla(M) = \mathbb{R}[h_F \mid F \in \mathscr{L}_M - \varnothing]/(I + J),$$

where $I = \langle h_a \mid a \in \mathfrak{A}_M\rangle$ and $J = \langle (h_F - h_{F\vee F'})(h_{F'} - h_{F \vee F'}) \mid F, F' \in \mathscr{L}_M - \varnothing\rangle$. We note that this presentation of J, appearing in [9], differs from that of [2].

In the *free matroid* on E, all subsets $A \subseteq E$ are flats. The Chow ring of the free matroid surjects onto $A^\bullet_\nabla(M)$, for any loopless matroid M on E, by $h_A \mapsto h_F$, where F is the closure in M of a given nonempty subset A of E. In what follows, we simply write $h_A(M)$, or h_A, for $h_{\mathrm{cl}_M(A)}(M)$. The simplicial presentation lends itself to the following combinatorial interpretation of the Chow ring of M. Let A be a nonempty subset of E with $\mathrm{rk}_M(A) > 1$ (if $\mathrm{rk}_M(A) = 1$, then $h_A = 0$ by definition). The simplicial generator h_A corresponds (via a combinatorial analogue of the cap product) to the *principal truncation* of M at the flat $F = \mathrm{cl}_M(A)$, denoted $T_F(M)$; this is the matroid with bases $B - f$ over all bases B of M which intersect F nontrivially, and over all $f \in B \cap F$ [2, Theorem 3.2.3]. Furthermore, the cap product allows for a bijection between the monomial basis for $A^c_\nabla(M)$, $\left\{h_{F_1}^{a_1} \cdots h_{F_k}^{a_k} \mid \sum a_i = c,\ \varnothing = F_0 \subsetneq F_1 \subsetneq \cdots \subsetneq F_k,\ 1 \le a_i < \mathrm{rk}_M(F_i) - \mathrm{rk}_M(F_{i-1})\right\}$, and a class of matroids called *loopless relative nested quotients of* M. We will forego the definition of these matroids until after defining their generalizations which appear in Theorem 2.1.

Let us recall some basic definitions and notations. For a matroid M on ground set E, we let $\mathcal{I}(M)$, $\mathcal{B}(M)$, and $\mathcal{C}(M)$ denote the collections of independent sets, bases, and circuits of M, respectively. The rank in M of a subset S of E is denoted by $\mathrm{rk}_M(S)$. The *uniform matroid* $U_{k,E}$ is the matroid whose bases are all of the k-subsets of E. The *dual* of M, denoted M^*, is the matroid on E whose bases are the complements of the bases of M. The *Bergman fan* Σ_M of M is the polyhedral fan in $\mathbb{R}^E/\langle e_E\rangle$ consisting of the cones $\mathrm{cone}\{e_F \mid F \in \mathcal{F}\}$ for each flag $\mathcal{F} = \{\varnothing \neq F_1 \subsetneq \cdots \subsetneq F_t \neq E\}$ of flats of M, where e_F denotes the indicator vector for F. For a matroid M of rank $d+1$, up to scaling, there is a unique d-dimensional Minkowski weight on Σ_M [1], which is known as the *Bergman class* Δ_M *of* M.

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For a nonempty subset S of E, let H_S denote the corank-1 matroid on E with collection of bases $\{E-s \mid s \in S\}$. It is well-known that the Bergman fans of corank-1 matroids are precisely the tropical hyperplanes centered at the origin. In [12], Speyer defined a notion of stable intersection for tropical linear spaces. This relates closely to matroid intersection, defined as follows. The *matroid intersection* of matroids M and N on a common ground set E, denoted $M \wedge N$, is the matroid whose spanning sets are the intersections of the spanning sets of M and N. It is noted in [7] that, as a special case of Theorem 4.11 in [12], the stable intersection of Bergman fans Σ_M and Σ_N is precisely $\Sigma_{M\wedge N}$ whenever $M \wedge N$ is loopless. A related notion is that of *matroid union*, defined in [10]: $M \vee N$ is the matroid whose independent sets are of the form $I \cup J$ for $I \in \mathcal{I}(M)$ and $J \in \mathcal{I}(N)$. We will make use of the fact that $M \wedge N$ can equivalently be defined as $(M^* \vee N^*)^*$.

We characterize the products of simplicial generators in $A^\bullet_\nabla(M)$ using the duals of certain Rado matroids. In order to define a Rado matroid, we first recall that a *transversal* of a collection $\{X_1, \dots, X_m\}$ of (not necessarily distinct) nonempty subsets of a finite set Y is a set $\{y_1, \dots, y_m\}$ of distinct elements in Y such that $y_i \in X_i$ for all i.

Theorem 1.1 (Rado's theorem [11])**.** *Let M be a matroid on Y, and let $\mathcal{X}$ be a collection of subsets $X_1, \dots, X_m$ of Y. There exists a transversal of $\mathcal{X}$ which is independent in M if and only if $\mathrm{rk}_M(\cup_{j\in J} X_j) \geq |J|$ for all $J \subseteq \{1, \dots, m\}$.*

Using Rado's theorem, it is not hard to show that the subsets of $\mathcal{X}$ which have independent transversals in M form the independent sets of a matroid on $\mathcal{X}$.

Definition 1.1. *For a matroid M on Y and a collection $\mathcal{X}$ of subsets $X_1, \dots, X_m$ of Y, the* Rado matroid induced by $\mathcal{X}$ and M *is a matroid with ground set $\mathcal{X}$. Its independent sets are given by the subcollections of $\mathcal{X}$ with a transversal in $\mathcal{I}(M)$.*

Abusing terminology slightly, given a bipartite graph H with ordered partition $(\mathcal{X}, Y)$, the subsets of $\mathcal{X}$ which can be matched to a set in $\mathcal{I}(M)$ form the independent sets of the *Rado matroid induced by H and M*, denoted $R_{H,M}$. The ground set of $R_{H,M}$ will always be the first part of the partition $(\mathcal{X}, Y)$ of H.

When M is the free matroid on Y, Rado's theorem specializes to Hall's theorem on transversals, and we obtain transversal matroids from Rado matroids.

2. Main result

We first introduce a class of graphs to be used in our characterization of products of simplicial generators in $A^\bullet_\nabla(M)$.

Definition 2.1. *Let $\mathcal{A}$ be a collection of nonempty subsets $A_1, \dots, A_m$ of a finite set E, and let $\hat{E}$ be a copy of E, $\hat{E} = \{\hat{e} \mid e \in E\}$. We define $G(\mathcal{A})$ to be the graph with bipartition $(E, \hat{E} \cup \mathcal{A})$ and edge set*

$$\{e\hat{e} \mid e \in E\} \cup \{eA \mid A \in \mathcal{A} \text{ and } e \in A\}.$$

Figure 2.1 depicts an example of such a graph. We remind the reader that $G(\mathcal{A})$ is *not* the graph typically used to represent the set system $\mathcal{A}$, which was denoted by H in Definition 1.1.

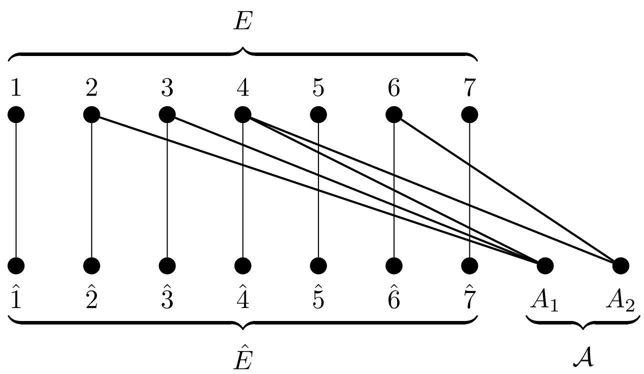

Figure 2.1: The graph $G(\mathcal{A})$, where $E = \{1, \dots, 7\}$, $\mathcal{A} = \{A_1, A_2\}$, $A_1 = \{2, 3, 4\}$, and $A_2 = \{4, 6\}$.

Theorem 2.1. *Let $\mathcal{A}$ be a collection of (not necessarily distinct) nonempty subsets $A_1, \dots, A_m$ of a finite set E, let M be a matroid on E, and let $G = G(\mathcal{A})$. We have*

$$M \wedge H_{A_1} \wedge \cdots \wedge H_{A_m} = (R_{G,N})^*,$$

where N is the matroid $\hat{M}^* \oplus U_{m,\mathcal{A}}$ on $\hat{E} \cup \mathcal{A}$, and $\hat{M}^*$ is a copy of M^* on $\hat{E}$.

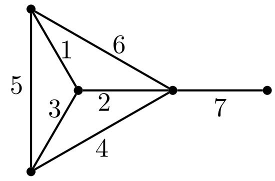

Figure 2.2: A graph whose edge set is the set E in Figure 2.1 and whose graphic matroid is not a strict gammoid.

Before proving Theorem 2.1, we note an important corollary. Namely, that the matroids $(R_{G,N})^*$ in Theorem 2.1, when they are loopless, are in natural bijection with the nonzero products of simplicial generators in $A^\bullet_\nabla(M)$. In fact, we can determine precisely which of these matroids are loopless using the *Dragon Hall-Rado condition of* M: for a matroid M on E, a collection $\mathcal{A}$ of subsets $A_1, \ldots, A_m$ of E is said to satisfy $\mathrm{DHR}(M)$ if $\mathrm{rk}_M(\cup_{j\in J} A_j) > |J|$ for any nonempty subset J of $\{1, \ldots, m\}$. Equivalently, $\mathcal{A}$ satisfies $\mathrm{DHR}(M)$ if and only if, for every $e \in E$, there is a transversal $I \subseteq E - e$ of $\mathcal{A}$ which is independent in M [2, Proposition 5.2.3]. That $R_{G(\mathcal{A}),N}$ has a coloop whenever $\mathcal{A}$ does not satisfy $\mathrm{DHR}(M)$ is clear from this equivalent definition, for if there is an element $e \in E$ which is in every independent transversal of $\mathcal{A}$, then it is in every basis of $R_{G(\mathcal{A}),N}$.

Corollary 2.1. *Let $\mathcal{A}$ be a collection of nonempty subsets $A_1, \ldots, A_m \subseteq E$, and let M be a loopless matroid on E. The product of simplicial generators $h_{A_1} \cdots h_{A_m}$ in $A^\bullet_\nabla(M)$ is the Bergman class of the matroid $(R_{G(\mathcal{A}),N})^*$ from Theorem 2.1 whenever $\mathcal{A}$ satisfies* $\mathrm{DHR}(M)$.

Proof of Corollary 2.1 assuming Theorem 2.1. We have noted the equivalence of a stable intersection of Bergman fans and the Bergman fan of a matroid intersection whenever the matroid intersection is loopless. We now show that, whenever $\mathcal{A}$ satisfies $\mathrm{DHR}(M)$, the matroid $M \wedge H_{A_1} \wedge \cdots \wedge H_{A_m}$ is loopless of rank $\mathrm{rk}_M(E) - m$. The result then follows form a direct application of Theorem 2.1.

We proceed by induction on $|\mathcal{A}|$. When $\mathcal{A} = \{A\}$, this is given by [2, Theorem 3.2.3]. Now, assuming $|\mathcal{A}| \geq 2$, let $A \in \mathcal{A}$, and let $e \in E$. Further, let $I \subseteq E - e$ be a transversal of $\mathcal{A}$ which is independent in M. Note that $E - I$ is a spanning set for M^*. If a denotes the representative for A in I, then $(E - I) \cup a$ spans $M^* \vee H^*_A$. Thus, $I - a$ is an independent set in $M \wedge H_A$. Since $I - a$ is a transversal of $\mathcal{A} - A$ which avoids the arbitrarily chosen element $e \in E$, we have that $\mathcal{A} - A$ satisfies $\mathrm{DHR}(M \wedge H_A)$. An application of the inductive hypothesis to the loopless matroid $M \wedge H_A$ of rank $\mathrm{rk}_M(E) - 1$ completes the proof. □

Proof of Theorem 2.1. We work by induction on m. Since $M \wedge H_{A_1} \wedge \cdots \wedge H_{A_m} = (M^* \vee H^*_{A_1} \vee \cdots \vee H^*_{A_m})^*$, it suffices to show that $M^* \vee H^*_{A_1} \vee \cdots \vee H^*_{A_m} = R_{G,N}$. The base case, $m = 0$, is trivial. Let $m \geq 1$, and assume that the result holds for the collection $\mathcal{A} - A_m$; that is,

$$R' := M^* \vee H^*_{A_1} \vee \cdots \vee H^*_{A_{m-1}}$$

is the Rado matroid on E induced by $G(\mathcal{A} - A_m)$ and $\hat{M}^* \oplus U_{m-1,\mathcal{A}-A_m}$. It suffices to show that the independent sets of $R' \vee H^*_{A_m}$ are precisely the independent sets of $R_{G,N}$.

First, suppose that a subset I of E is independent in $R' \vee H^*_{A_m}$. We will show that I is matched in G to an independent set in the matroid $\hat{M}^* \oplus U_{m,\mathcal{A}}$ on N. By definition, $I = J \cup K$, where $J \in \mathcal{I}(R')$ and $K \in \mathcal{I}(H^*_{A_m}) = \mathcal{I}(U_{1,A_m})$. If $K \subseteq J$, then I is independent in R', and thus I is independent in $R_{G,N}$. Otherwise, we have $K = \{a\}$ for some $a \notin J$. Take a matching from J to an independent set of R' and add the edge aA_m, which is clearly disjoint from the others, to obtain a matching from I to an independent set in $R_{G,N}$.

Second, suppose that $I \subseteq E$ is matched in G to a subset L of $\hat{E} \cup \mathcal{A}$ which is independent in $\hat{M}^* \oplus U_{m,\mathcal{A}}$. If $A_m \notin L$, then I is matched in G to an independent set in $\hat{M}^* \oplus U_{m-1,\mathcal{A}-A_m}$ on $\hat{E} \cup \{A_1, \ldots, A_{m-1}\}$; that is, $I \in \mathcal{I}(R')$, and thus $I \in \mathcal{I}(R' \vee H^*_{A_m})$. Otherwise, if $A_m \in L$, then there exists some a in $A_m \cap I$ such that $I - a$ is matched to an independent set in $\hat{M}^* \oplus U_{m-1,\mathcal{A}-A_m}$ on $\hat{E} \cup \{A_1, \ldots, A_{m-1}\}$. Thus, $I - a \in \mathcal{I}(R')$, which implies that $I \in \mathcal{I}(R' \vee H^*_{A_m})$. We have shown that $\mathcal{I}(R' \vee H^*_{A_m}) = \mathcal{I}(R_{G,N})$, which completes the proof. □

Example 2.1. *Let us again consider the graph $G = G(\mathcal{A})$ in Figure 2.1. Let M be the graphic matroid for the graph with edge set E depicted in Figure 2.2. The matroid M has rank 4. Letting $N = \hat{M}^* \oplus U_{2,\mathcal{A}}$ and $R = R_{G,N}$, we have $M \wedge H_{A_1} \wedge H_{A_2} = R^*$ by Theorem 2.1. The matroid R^* is a rank-2 matroid with set of bases $\{17, 27, 37, 47, 57, 67\}$. For instance, $\{1, 2, 3, 5, 6\} \subset E$ is matched in G to $\{\hat{1}, \hat{3}, \hat{5}, A_1, A_2\} \in \mathcal{B}(M^* \oplus U_{2,\mathcal{A}})$, and so $\{4, 7\} \in \mathcal{B}(R^*)$.*

The duals of transversal matroids are known as *strict gammoids*. We refer to the dual of a Rado matroid as a *coRado matroid*. As we noted earlier, the Bergman fans of corank-1 matroids are precisely the tropical hyperplanes centered at the origin. Thus, Theorem 2.1 implies that the stable intersection of a Bergman fan Σ_M with a collection of tropical hyperplanes centered at the origin is the Bergman fan of a coRado matroid $(R_{G,N})^*$. Letting M be the free matroid, we recover a special case of a theorem of Fink and Olarte.

Corollary 2.2 (see [6])**.** *A matroid is a strict gammoid if and only if its Bergman fan is a stable intersection of tropical hyperplanes centered at the origin.*

Theorem 7.5 of [6] states, more generally, that a valuated matroid is a valuated strict gammoid if and only if its associated tropical linear space is a stable intersection of tropical hyperplanes; Corollary 2.2 is the special case in which the valuations are trivial. The graphic matroid for the graph in Figure 2.2, however, is not a strict gammoid, and thus the Bergman fan of the coRado matroid R^* in Example 2.1 is not a stable intersection of tropical hyperplanes.

We now return to the loopless relative nested quotients of a matroid M, which are shown in [2] to be in correspondence with the monomial bases for the graded pieces of $A^{\bullet}_{\nabla}(M)$. First, we note that any principal truncation $T_F(M)$ is given by the dual of the Rado matroid induced by the graph $G(\{F\})$ and the matroid $\hat{M}^* \oplus U_{1,\{F\}}$ on $\hat{E} \sqcup \{F\}$. Now, we recall that the monomial basis for the graded piece of degree c, $A^c_{\nabla}(M)$, is the set of products $h_{F_1}^{a_1} \cdots h_{F_m}^{a_m}$ of simplicial generators corresponding to nested nonempty flats $F_1, \ldots, F_m \in \mathscr{L}_M$, with each $1 \le a_i < \mathrm{rk}_M(F_i) - \mathrm{rk}_M(F_{i-1})$ and $\sum a_i = c$. The coRado matroids in Theorem 2.1 provide a new definition for the *relative nested quotients of* M: they are matroids of the form $(R_{G(\mathcal{A}),N})^*$, where $\mathcal{A}$ is a multiset of nested flats as described above, with a_i copies of F_i for each i.

The graphs $G(\mathcal{A})$ in Definition 2.1 and the Rado matroids associated to them in Theorem 2.1 can also be used to provide an alternate proof of the Dragon Hall-Rado theorem of [2]. Coincidentally, Larson also obtained an alternate proof in [8], posted the day before the first preprint of this paper was made available.

Corollary 2.3 (Dragon Hall-Rado theorem [2])**.** *Let $A_1, \ldots, A_d$ be nonempty subsets of a finite set E, and let M be a loopless matroid on E of rank $d+1$. We have $M \wedge H_{A_1} \wedge \cdots \wedge H_{A_d} = U_{1,E}$ if and only if $\{A_1, \ldots, A_d\}$ satisfies* $\mathrm{DHR}(M)$.

Proof. Let $\mathcal{A} = \{A_1, \ldots, A_d\}$, let $G = G(\mathcal{A})$, and let $R = R_{G,N}$, where N is the matroid $\hat{M}^* \oplus U_{d,\mathcal{A}}$ on $\hat{E} \cup \mathcal{A}$ (as in Theorem 2.1). We recall that $\mathcal{A}$ satisfies $\mathrm{DHR}(M)$ if and only if, for any $e \in E$, there is a transversal I of $\mathcal{A}$ such that $e \notin I$ and $I \in \mathcal{I}(M)$. Thus, it suffices to check that $R = U_{|E|-1,E}$ if and only if, for any $e \in E$, $\mathcal{A}$ is matched in G to an independent set in M which does not contain e.

To prove sufficiency, we recall from the proof of Corollary 2.1 that, when $\mathcal{A}$ satisfies $\mathrm{DHR}(M)$, the coRado matroid R^* is loopless of rank $d+1-d$. To prove necessity, suppose that $R = U_{|E|-1,E}$, and let $e \in E$ be arbitrary. Since $E - e$ is a basis of R, it is matched in G to a basis $\hat{B}^*$ of $\hat{M}^*$ (of cardinality $|E| - d - 1$) and to each of the vertices $A_1, \ldots, A_d \in V(G)$. Letting I be the set of vertices matched to $A_1, \ldots, A_d$, we see that I is independent in M and does not contain e. This completes the proof. □

Eur and Larson [4] generalized the simplicial presentation for the Chow ring of a matroid to augmented Chow rings of polymatroids. We expect that Theorem 2.1 generalizes to the case of polymatroids as well. We welcome interested researchers to explore this direction.

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References

[1] K. Adiprasito, J. Huh, E. Katz. Hodge theory for combinatorial geometries, *Ann. Math.* **188(2)** (2018) 381–452.
[2] S. Backman, C. Eur, C. Simpson, Simplicial generation of Chow rings of matroids, *J. Eur. Math. Soc.*, In press, DOI: 10.4171/JEMS/1350.
[3] C. De Concini, C. Procesi, Wonderful models of subspace arrangements, *Selecta Math.* **1** (1995) 459–494.
[4] C. Eur, M. Larson, Intersection theory of polymatroids, *Int. Math. Res. Not. IMRN* **2024(5)** (2024) 4207–4241.
[5] E. M. Feichtner, S. Yuzvinsky, Chow rings of toric varieties defined by atomic lattices, *Invent. Math.* **155(3)** (2004) 515–536.
[6] A. Fink, J. A. Olarte, Presentations of transversal valuated matroids, *J. Lond. Math. Soc.* **105(1)** (2022) 24–62.
[7] S. Hampe, The intersection ring of matroids, *J. Combin. Theory Ser. B* **122** (2017) 578–614.
[8] M. Larson, Straightening laws for Chow rings of matroids, *arXiv*:2402.03444 [math.CO], (2024).
[9] M. Larson, S. Li, S. Payne, N. Proudfoot, K-rings of wonderful varieties and matroids, *Adv. Math.* **441** (2024) #109554.
[10] C. S. J. A. Nash-Williams, An application of matroids to graph theory, In: *Theory of Graphs, International Symposium*, Rome, 1966, 263–265.
[11] R. Rado, A theorem on independence relations, *Q. J. Math.* **13** (1942) 83–89.
[12] D. E. Speyer, Tropical linear spaces, *SIAM J. Discrete Math.* **22(4)** (2008) 1527–1558.
[13] S. Yuzvinsky, Small rational model of subspace complement, *Trans. Amer. Math. Soc.* **354(5)** (2002) 1921–1945.