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*Research Article*

# The shortest cycle having the maximal number of coalition graphs

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**Abstract**

A coalition in a graph $G$ with a vertex set $V$ consists of two disjoint sets $V_1, V_2 \subset V$, such that neither $V_1$ nor $V_2$ is a dominating set, but the union $V_1 \cup V_2$ is a dominating set in $G$. A partition of $V$ is called a coalition partition $\pi$ if every non-dominating set of $\pi$ is a member of a coalition and every dominating set is a single-vertex set. Every coalition partition generates its coalition graph. The vertices of the coalition graph correspond one-to-one with the partition sets and two vertices are adjacent if and only if their corresponding sets form a coalition. In the paper [T. W. Haynes, J. T. Hedetniemi, S. T. Hedetniemi, A. A. McRae, R. Mohan, *Discuss. Math. Graph Theory* **43** (2023) 931–946], the authors proved that partition coalitions of cycles can generate only 27 coalition graphs and asked about the shortest cycle having the maximum number of coalition graphs. In this paper, we show that $C_{15}$ is the shortest graph having this property.

**Keywords:** cycle; coalition partition; coalition number.

**2020 Mathematics Subject Classification:** 05C69, 68Q25.

## 1. Introduction

Let $G$ be a simple graph. We denote the vertex set and edge set of $G$ as $V(G)$ and $E(G)$, respectively. The corona $G \circ K_1$ of a graph $G$ is the graph obtained by attaching a pendant edge at each vertex of $G$. A set $S \subseteq V(G)$ is said to be a dominating set if every $v \in V(G) \setminus S$ is adjacent to some vertex in $S$. The domination number $\gamma(G)$ is the minimum size of a dominating set of $G$. For a detailed treatment of domination theory, we refer the reader to [13–15].

In [8], Haynes et al. introduced the concept of coalitions in graphs. This concept has been studied in several publications; for instance, see [4–7, 9–12]. Let $V_1$ and $V_2$ be two disjoint subsets of $V(G)$. They form a coalition if neither of them is a dominating set, but their union $V_1 \cup V_2$ is. A vertex partition $\pi = \{V_1, V_2, \ldots, V_k\}$ of $V(G)$ is called a coalition partition if every non-dominating set of $\pi$ is a member of a coalition and every dominating set is a single-vertex set. The maximum cardinality of a coalition partition is the coalition number of a graph $G$, and is denoted by $C(G)$. Various types of this concept have been studied in [1–3].

The coalition graph (c-graph), denoted by $CG(G, \pi)$, is obtained by associating the partition sets of $\pi$ with vertices, and two vertices of $CG(G, \pi)$ are adjacent if and only if the corresponding sets form a coalition in $G$. In [10], the authors showed that there are only 18 c-graphs of paths. A path is called coalition universal if its coalition partitions define all 18 c-graphs. Henning et al. [5] proved that there are no universal coalition paths and $P_{10}$ is the shortest path that defines the maximal number of c-graphs, which is 15. They determined the number of c-graphs of $P_k$ for a given positive integer $k$.

Haynes et al. [10] showed that there are exactly 27 graphs of order at most 6 that can be c-graphs of cycles (see Figure 1.1): $K_2$, $P_3$, $P_4$, $P_5$, $2K_2$, $K_3$, $C_4$, $C_5$, $K_{1,3}$, the graphs $F_1$ and $F_2$, the diamond $K_4 - e$, $K_4$, $P_2 \cup P_3$, $K_2 \cup K_3$, the house graph $H$, the double stars $S(1, 2)$ and $S(2, 2)$, the bull graph $B$, the graphs $H_1$, $H_2$, and $H_3$, $3K_2$, $K_2 \cup P_4$, the corona $P_3 \circ K_1$, and the corona $K_3 \circ K_1$. A cycle is called coalition universal if its coalition partitions define all 27 c-graphs.

In this paper, we answer the following question posed by Haynes et al. in [10].

**Question 1.1.** *Does there exist a universal coalition cycle, that is, a cycle $C_k$ on which all 27 coalition graphs can be defined? If so, what is the smallest such integer $k$?*

Since the c-graph $\overline{K}_3$ has no edges, every set of the corresponding coalition partition should be a dominating set, that is, by definition, every set should be a single-vertex set. Obviously, it is only possible for the cycle of order 3, and consequently, $K_3$ is exclusively a c-graph for $C_3$. Therefore, there are no universal coalition cycles, and we reformulate Question 1.1 as follows.

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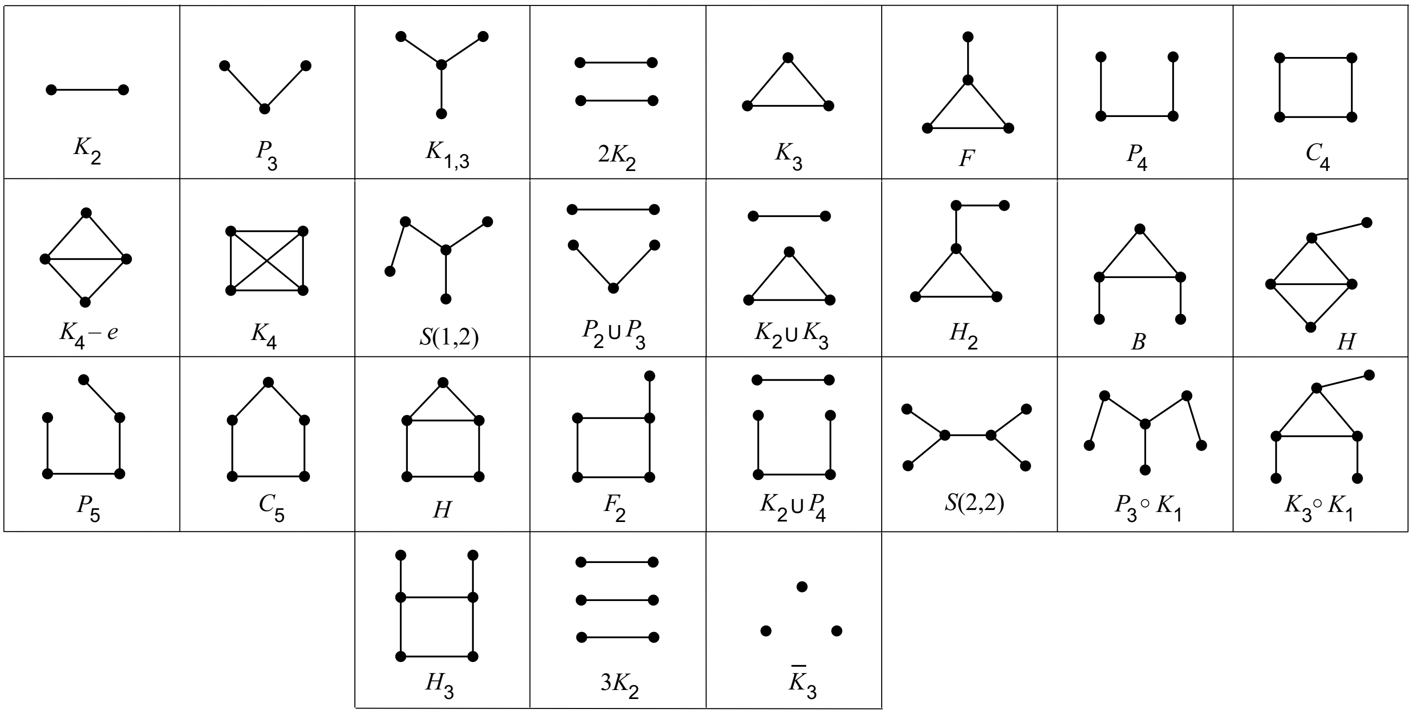


**Figure 1.1:** Coalition graphs defined by cycles.

**Question 1.2.** *Does there exist a cycle $C_k$ on which all 26 coalition graphs can be defined? If so, what is the smallest such integer $k$?*

## 2. Coalition graphs defined by $C_{15}$

It is known that the Stirling numbers of the second kind, $s(n,k)$, count partitions of a set of cardinality $n$ into $k$ nonempty subsets. To compute the numbers of all c-graphs for $C_{15}$, it is sufficient to check necessary properties of $s(15,k)_{k=2..6} = \{16383, 2375101, 42355950, 210766920, 420693273\}$ partitions [16]. The results of computer enumeration of the number of c-graphs of cycles up to 15 vertices are given in Table 1. According to these results, the graph $C_{15}$ is the shortest cycle generating all 26 c-graphs.

The c-graphs defined by $C_{15}$ and its coalition partitions are shown in Table 2. We assume that the vertices of $C_{15}$ are given consecutive positive integers. The number before a coalition set represents a vertex of a c-graph.

## 3. Coalition graphs of cycles of order less than 15

In this section, we prove that the number of c-graphs for cycles $C_k$, $k \leq 14$, is less than 26. In order to achieve this goal, it suffices to show that there is a c-graph $G$ of Figure 1.1, which is not derived by the cycle $C_k$, where $k \leq 14$. We first present an observation, which will be useful for our proofs.

**Observation 3.1.** **(i).** *Let $\pi(G) = \{V_1, V_2, \ldots, V_k\}$ be a coalition partition of a graph $G$. Then $|V_i \cup V_j| \geq \gamma(G)$ for any two coalition sets $V_i$ and $V_j$ of $\pi(G)$.*

**(ii).** *Let $S_1$ and $S_2$ be the minimal dominating sets (of cardinality $k$) in cycle $C_{3k}$. Then the sets $S_1$ and $S_2$ are always disjoint.*

**(iii).** *The cycle $C_{3k}$, $k \geq 1$, has three disjoint minimal dominating sets of order $k$. After removing two minimal dominating sets from the cycle, the remaining vertices form a minimal dominating set.*

Now, we prove the following result:

**Theorem 3.1.** *The cycle $C_k$ does not define the c-graph $C_5$, where $k \in \{6, 7, 9\}$.*

**Table 1:** Number of coalition graphs of $C_k$, $k \leq 15$.

| $CG(C_k)$ | $C_4$ | $C_5$ | $C_6$ | $C_7$ | $C_8$ | $C_9$ | $C_{10}$ | $C_{11}$ | $C_{12}$ | $C_{13}$ | $C_{14}$ | $C_{15}$ |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $K_2$ | . | . | 3 | 7 | 20 | 54 | 130 | 308 | 711 | 1599 | 3542 | 7740 |
| $P_3$ | . | . | 18 | 56 | 224 | 873 | 2940 | 9878 | 32400 | 103220 | 324632 | 1007625 |
| $K_3$ | . | 5 | 8 | 49 | 184 | 573 | 1910 | 5951 | 17856 | 53235 | 155526 | 448285 |
| $2K_2$ | . | . | 3 | 7 | 38 | 243 | 1025 | 4565 | 19890 | 80184 | 317891 | 1232520 |
| $K_{1,3}$ | . | . | 6 | 14 | 96 | 450 | 1760 | 7623 | 29668 | 111176 | 414946 | 1494235 |
| $P_4$ | . | . | 18 | 56 | 288 | 1278 | 4760 | 17930 | 65070 | 224952 | 767760 | 2572200 |
| $F_1$ | . | 5 | . | 42 | 204 | 639 | 2490 | 8525 | 27138 | 87256 | 268926 | 813665 |
| $C_4$ | . | . | 3 | 14 | 48 | 171 | 560 | 1716 | 5166 | 15210 | 43820 | 124590 |
| $K_4 - e$ | . | . | 6 | 28 | 40 | 198 | 650 | 1287 | 4116 | 11440 | 25928 | 70995 |
| $P_2 \cup P_3$ | . | . | 6 | . | 40 | 180 | 425 | 2585 | 8388 | 25064 | 102816 | 307020 |
| $S(1,2)$ | . | . | 6 | 14 | 48 | 342 | 920 | 3168 | 12084 | 34229 | 112742 | 359550 |
| $P_5$ | . | . | . | 7 | 48 | 72 | 260 | 1232 | 1950 | 5967 | 21497 | 38445 |
| $B$ | . | . | . | 7 | 16 | 27 | 150 | 451 | 972 | 3081 | 9289 | 21525 |
| $K_4$ | 1 | . | . | 7 | 7 | . | 50 | 77 | 59 | 364 | 672 | 865 |
| $F_2$ | . | . | . | . | 8 | 9 | 10 | 154 | 216 | 364 | 2198 | 3645 |
| $S(2,2)$ | . | . | . | . | 4 | . | 15 | 110 | 114 | 546 | 2324 | 3960 |
| $C_5$ | . | 1 | . | . | 8 | . | 3 | 44 | . | 26 | 210 | 16 |
| $K_2 \cup K_3$ | . | . | . | . | . | 18 | . | . | 312 | . | . | 3690 |
| $H_2$ | . | . | . | . | . | 18 | . | . | 216 | . | . | 2040 |
| $H_1$ | . | . | . | . | . | . | . | . | 48 | . | . | 480 |
| $H$ | . | . | . | . | . | . | . | . | 12 | . | . | 90 |
| $3K_2$ | . | . | 1 | . | . | 9 | . | . | 123 | . | . | 1540 |
| $K_2 \cup P_4$ | . | . | . | . | . | 9 | . | . | 120 | . | . | 1095 |
| $P_3 \circ K_1$ | . | . | . | . | . | 9 | . | . | 84 | . | . | 615 |
| $K_3 \circ K_1$ | . | . | . | . | . | . | . | . | 4 | . | . | 35 |
| $H_3$ | . | . | . | . | . | . | . | . | 12 | . | . | 90 |

**Table 2:** Coalition partitions for all coalition graphs of $C_{15}$.

| $CG(C_{15})$ | coalition partitions for $CG(C_{15})$ | edges of $CG(C_{15})$ |
|---|---|---|
| $K_2$ | 1:(1, 2,...,12) 2:(13, 14, 15) | 1-2 |
| $P_3$ | 1:(1, 2,...,12) 2:(13, 15) 3:(14) | 1-2; 1-3 |
| $K_3$ | 1:(1, 2, 5, 7, 9, 11) 2:(3, 13, 14) 3:(4, 6, 8, 10, 12, 15) | 1-2,3; 2-3 |
| $K_{1,3}$ | 1:(1, 2,...,12) 2:(13) 3:(14) 4:(15) | 1-2; 1-3; 1-4 |
| $2K_2$ | 1:(1, 2, 5, 10, 11) 2:(3, 12) 3:(4, 8, 13, 14) 4:(6, 7, 9, 15) | 1-3; 2-4 |
| $P_4$ | 1:(1, 2, 5, 10, 11, 14) 2:(3, 12, 15) 3:(4, 8) 4:(6, 7, 9, 13) | 1-3,4:2-4 |
| $F_1$ | 1:(1, 2, 5, 8, 10, 11) 2:(3, 12, 15) 3:(4, 14) 4:(6, 7, 9, 13) | 1-2,3,4; 2-4 |
| $C_4$ | 1:(1, 2, 5, 8, 11) 2:(3, 9, 15) 3:(4, 7, 12, 14) 4:(6, 10, 13) | 1-3,4; 2-3,4 |
| $K_4 - e$ | 1:(1, 2, 5, 8, 11) 2:(3, 9, 12, 15) 3:(4, 7, 14) 4:(6, 10, 13) | 1-2,3,4; 2-3,4 |
| $K_4$ | 1:(1, 4, 7, 10) 2:(2, 8, 11, 14) 3:(3, 6, 13) 4:(5, 9, 12, 15) | 1-2,3,4; 2-3,4; 3-4 |
| $P_2 \cup P_3$ | 1:(1, 2, 5, 8, 11) 2:(3, 15) 3:(4, 14) 4:(6, 9, 12) 5:(7, 10, 13) | 1-3,5; 2-4 |
| $S(1,2)$ | 1:(1, 2, 5, 8, 11) 2:(3) 3:(4, 14) 4:(6, 9, 12, 15) 5:(7, 10, 13) | 1-3,4,5; 2-4 |
| $P_5$ | 1:(1, 2, 5, 8, 13) 2:(3, 9, 14, 15) 3:(4, 7, 12) 4:(6, 11) 5:(10) | 1-4,5; 2-3,4 |
| $B$ | 1:(1, 2, 5, 8) 2:(3, 9, 12, 15) 3:(4, 7) 4:(6, 11, 14) 5:(10, 13) | 1-2,4,5; 2-3,4 |
| $F_2$ | 1:(1, 2, 5, 8, 14) 2:(3, 15) 3:(4, 7, 10, 13) 4:(6, 9, 12) 5:(11) | 1-3,4,5; 2-3,4 |
| $S(2,2)$ | 1:(1, 2, 5, 8, 13, 14) 2:(3, 9, 12, 15) 3:(4, 7) 4:(6) 5:(10) 6:(11) | 1-2,5,6; 2-3,4 |
| $C_5$ | 1:(1, 4, 13) 2:(2, 7, 10) 3:(3, 8, 15) 4:(5, 11, 14) 5:(6, 9, 12) | 1-2,5; 2-4; 3-4,5 |
| $K_2 \cup K_3$ | 1:(1, 4, 7, 10) 2:(2, 14) 3:(3, 13) 4:(5, 8, 11) 5:(6, 9, 12, 15) | 1-3,5; 2-4; 3-5 |
| $H_2$ | 1:(1, 4, 7, 10) 2:(2, 13) 3:(3, 15) 4:(5, 8, 11, 14) 5:(6, 9, 12) | 1-2,4; 2-4; 3-4,5 |
| $H_1$ | 1:(1, 4, 7, 10) 2:(2, 13) 3:(3, 12, 15) 4:(5, 8, 11, 14) 5:(6, 9) | 1-2,3,4; 2-4; 3-4,5 |
| $H$ | 1:(1, 4, 7, 13) 2:(2, 10) 3:(3, 15) 4:(5, 8, 11, 14) 5:(6, 9, 12) | 1-2,4,5; 2-4; 3-4,5 |
| $3K_2$ | 1:(1, 4, 7, 10) 2:(2, 14) 3:(3, 15) 4:(5, 8, 11) 5:(6, 9, 12) 6:(13) | 1-6; 2-4; 3-5 |
| $K_2 \cup P_4$ | 1:(1, 4, 7, 10) 2:(2, 14) 3:(3) 4:(5, 8, 11) 5:(6, 9, 12, 15) 6:(13) | 1-5,6; 2-4; 3-5 |
| $P_3 \circ K_1$ | 1:(1, 4, 7, 10) 2:(2) 3:(3, 15) 4:(5, 8, 11, 14) 5:(6, 9, 12) 6:(13) | 1-4,6; 2-4; 3-4,5 |
| $K_3 \circ K_1$ | 1:(1, 4, 7, 10) 2:(2) 3:(3, 12, 15) 4:(5, 8, 11, 14) 5:(6, 9) 6:(13) | 1-3,4,6; 2-4; 3-4,5 |
| $H_3$ | 1:(1, 4, 7, 13) 2:(2) 3:(3, 15) 4:(5, 8, 11, 14) 5:(6, 9, 12) 6:(10) | 1-4,5,6; 2-4; 3-4,5 |

**Proof.** The vertex sets of cycles $C_6$, $C_7$, and $C_9$ can be partitioned into five subsets, whose cardinalities are listed below in descending order:

1). 2 1 1 1 1
2). 3 1 1 1 1    3). 2 2 1 1 1
4). 5 1 1 1 1    5). 4 2 1 1 1    6). 3 3 1 1 1    7). 3 2 2 1 1    8). 2 2 2 2 1

Then the cycle $C_5$ can potentially be configured using these partitions as shown in Figure 3.1. The brackets represent the vertices of $C_5$, and the number inside them indicates the cardinality of the corresponding non-dominating set. We demonstrate that these graphs cannot be the c-graph $C_5$ defined by the cycle $C_k$ for $k \in \{6, 7, 9\}$.

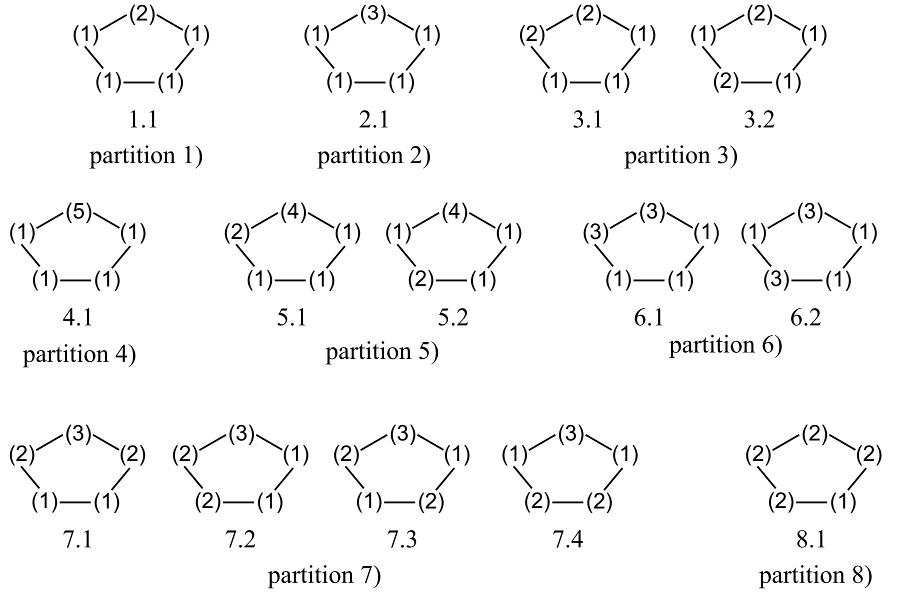


**Figure 3.1:** All possible c-graphs $C_5$ derived by vertex partitions of $C_6, C_7$, and $C_9$.

**Case 1.** Assume that the cycles 2.1, 3.1, 3.2, 4.1, 5.1, 5.2, 6.1, 6.2, 7.1, and 7.2 are c-graphs $C_5$. It can be seen that each of these graphs contains a dominating set with cardinality less than $\gamma(C_7)$ or $\gamma(C_9)$. This is impossible due to part $(i)$ of Observation 3.1.

**Case 2.** Assume that the cycles 1.1, 7.3, and 8.1 are c-graphs $C_5$. Then each of these graphs contains two minimal dominating sets of order 2 or 3 having a vertex in common, contradicting part $(ii)$ of Observation 3.1.

**Case 3.** Assume that the cycle 7.4 is a c-graph $C_5$. This graph has two disjoint minimal dominating sets of order 3, but the remaining set is not a minimal dominating set of order 3. This contradicts part $(iii)$ of Observation 3.1. □

Now, we demonstrate that the c-graph $3K_2$ is not derived by the cycle $C_k$, where $k \in \{8, 10, 11, 13, 14\}$.

**Theorem 3.2.** *The cycle $C_k$ does not define the c-graph $3K_2$, where $k \in \{8, 10, 11, 13, 14\}$.*

**Proof.** Let $\pi = \{V_1, V_2, \ldots, V_6\}$ is a coalition partition of cycle $C_k$, where $k \in \{8, 10, 11, 13, 14\}$. Without loss of generality, assume that sets $V_{2i-1} \cup V_{2i}$, $i \in \{1, 2, 3\}$, form coalitions in $\pi$.

(i). Let $\pi$ be the coalition partition of the cycle $C_8$. Assume that $|V_i| = 3$ for exactly one value of $i \in \{1, 2, \ldots, 6\}$ or $|V_i| = 2$ for exactly two values of $i \in \{1, 2, \ldots, 6\}$ (in both cases, $|V_j| = 1$ for all $j \neq i$). Then there exist $i$ and $j$ such that $|V_i \cup V_j| = 2$, contradicting part $(i)$ of Observation 3.1.

(ii). Let $\pi$ be a coalition partition of cycle $C_{10}$ or $C_{11}$. Since $\gamma(C_{10}) = \gamma(C_{11}) = 4$, $|V_{2i-1} \cup V_{2i}| \geq 4$ for all $i \in \{1, 2, 3\}$. So, $\sum_{i=1}^{6} |V_i| \geq 12$. Recall that $\pi$ is a vertex partition of the cycles. Then $\sum_{i=1}^{6} |V_i| \in \{10, 11\}$. This implies a contradiction.

(iii). Let $\pi$ be a coalition partition of cycle $C_{13}$ or $C_{14}$. Since $\gamma(C_{13}) = \gamma(C_{14}) = 5$, we can write $|V_{2i-1} \cup V_{2i}| \geq 5$ for all $i \in \{1, 2, 3\}$. Thus, $\sum_{i=1}^{6} |V_i| \geq 15$. Since $\pi$ is a vertex partition of the cycles, $\sum_{i=1}^{6} |V_i| \in \{13, 14\}$. This leads to a contradiction. □

**Theorem 3.3.** *The cycle $C_{12}$ does not define the c-graph $C_5$.*

**Proof.** The vertex sets of the cycle $C_{12}$ can be partitioned into five subsets, whose cardinalities are listed below in descending order:

| | | | | |
|---|---|---|---|---|
| 1). 8 1 1 1 1 | 4). 6 2 2 1 1 | 7). 5 2 2 2 1 | 10). 4 3 2 2 1 | 13). 3 3 2 2 2. |
| 2). 7 2 1 1 1 | 5). 5 4 1 1 1 | 8). 4 4 2 1 1 | 11). 4 2 2 2 2 | |
| 3). 6 3 1 1 1 | 6). 5 3 2 1 1 | 9). 4 3 3 1 1 | 12). 3 3 3 2 1 | |

All possible configurations of a cycle $C_5$ defined by these partitions are shown in Figure 3.2. We demonstrate that these graphs cannot be the c-graph $C_5$ defined the cycle $C_{12}$.

**Case 1.** Assume that all cycles associated with partitions 1) through 8) and the cycles 9.1, 9.2, 10.1 – 10.4, and 12.1 are c-graphs $C_5$. It can be seen that each of these graphs contains a dominating set with cardinality less than $\gamma(C_{12})$. This is impossible due to part $(i)$ of Observation 3.1.

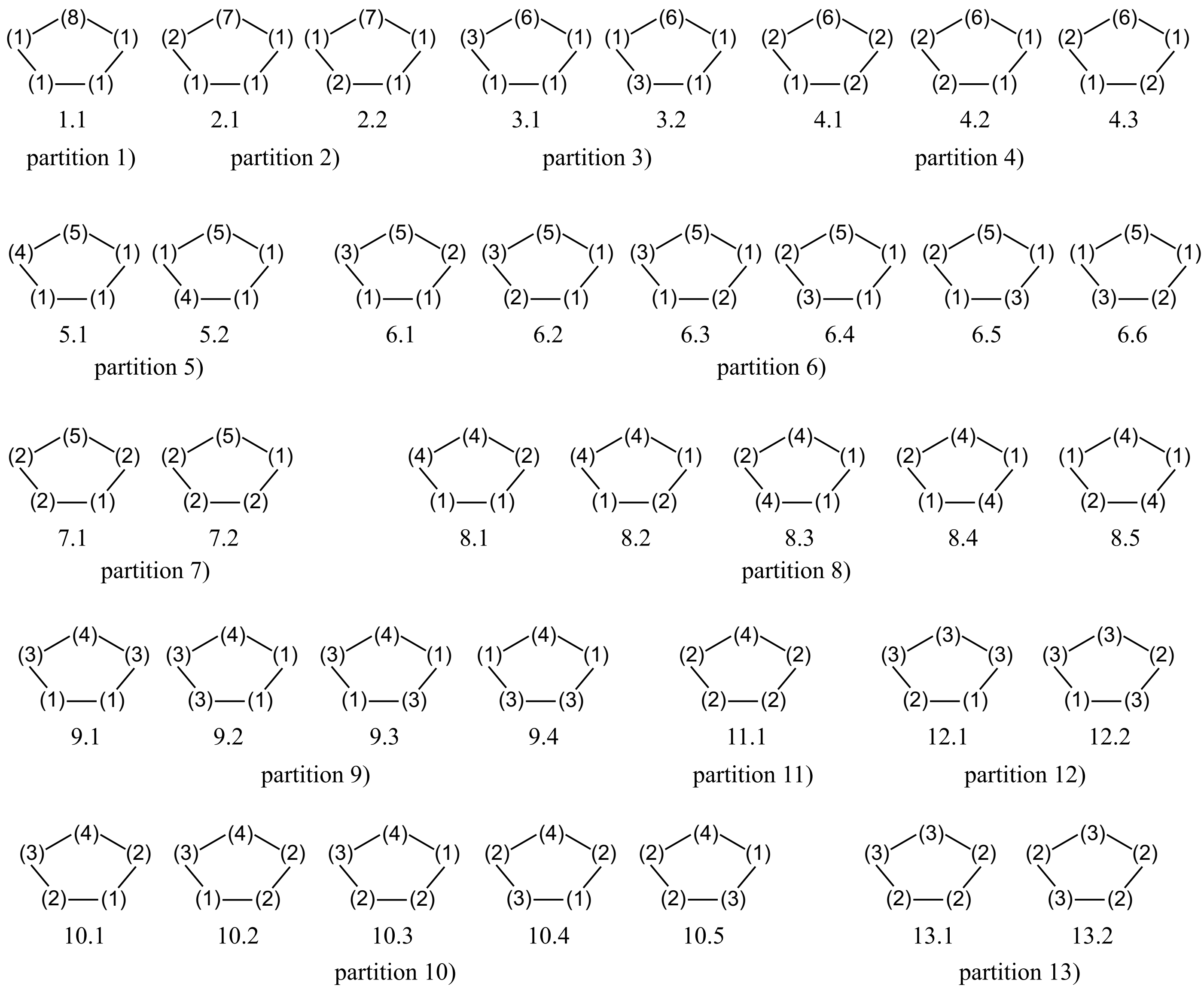


**Figure 3.2:** All possible c-graphs $C_5$ derived by vertex partitions of $C_{12}$.

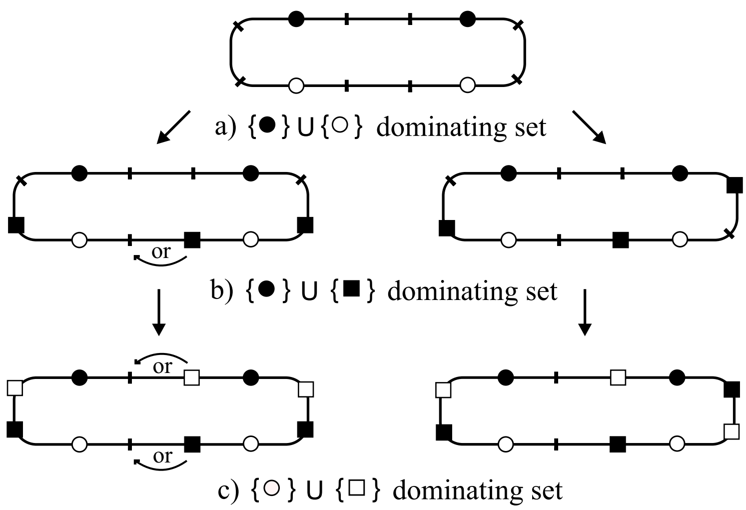


**Figure 3.3:** Placement coalitions in $C_{12}$ generated by cycle 13.2.

**Case 2.** Assume that the cycles 9.3, 11.1, 12.2, and 13.1 are c-graphs $C_5$. Then each of these graph contains two minimal dominating sets of order 4 having a vertex in common, contradicting part $(ii)$ of Observation 3.1.

**Case 3.** Assume that the cycles 9.4 and 10.5 are c-graphs $C_5$. It can be observed that each of these graphs has two disjoint minimal dominating sets of order 4, but the remaining set is not a minimal dominating set of order 4. This contradicts part $(iii)$ of Observation 3.1.

**Case 4.** Assume that the cycle 13.2 is a c-graph $C_5$. We consider the sequential placement of possible coalitions in the graph $C_{12}$. The vertices of two non-dominating sets, corresponding to the coalition (2)–(2), are marked with black and white circles. In addition, black and white squares are used to indicate the vertices of sets with cardinality 3 in the path (3)–(2)–(2)–(3) in such a way that the vertices associated with coalitions (2)–(3) have identical colors.

The unique (up to symmetry) placement of a possible coalition (2)–(2) is depicted in Figure 3.3a. The black coalition (2)–(3) can only be positioned in the manners illustrated in Figure 3.3b. Next, the white coalition (2)–(3) can only be located as shown in Figure 3.3c. It can be verified that the union of two sets of cardinality 3 (black and white squares) is a dominating set of $C_{12}$, which means that a new edge should connect the corresponding vertices in the c-graph. Then there exist two vertices having degree greater than 2, which is impossible. Therefore, the graph $C_{12}$ cannot define the c-graph $C_5$. □

## 4. Conclusion

We have shown that the cycle $C_{15}$ is the shortest graph that defines all c-graphs of Figure 1.1, except $\overline{K}_3$. This raises the following problem:

**Problem 4.1.** *Describe all universal cycles.*

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