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Research Article

Inversion sequences and signed permutations

Toufik Mansour*, Amir Safadi

Department of Mathematics, University of Haifa, 3103301 Haifa, Israel

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Abstract

A signed inversion sequence of length n is a sequence of integers $e = e_1 \cdots e_n$, where $e_{i+1} \in \{0, \bar{0}, 1, \bar{1}, \ldots, i, \bar{i}\}$ for every $i \in \{0, 1, \ldots, n-1\}$. For a set of signed patterns B, let $\bar{\mathcal{I}}_n(B)$ be the set of signed inversion sequences of length n that avoid all the signed patterns from B. We say that two sets of signed patterns B and C are Wilf-equivalent if $|\bar{\mathcal{I}}_n(B)| = |\bar{\mathcal{I}}_n(C)|$ for every $n \geq 0$. In this paper, by generating trees, we show that the number of Wilf-equivalences among singles of a length-2 signed pattern is 3 and the number of Wilf-equivalences among pairs of a length-2 signed patterns is 30.

Keywords: inversion sequences; signed inversion sequences; Wilf-equivalences; generating trees.

2020 Mathematics Subject Classification: 05A05, 05A15.

1. Introduction

The goal of this paper is to give analogies of enumerative results on inversion sequences related to permutations characterized by pattern-avoidance in the inversion sequences related to signed permutations. Let B_n be the hyperoctahedral group, which is the natural analogue of the symmetric group S_n. We identify classes of restricted inversion sequences of elements of B_n with enumerative properties analogous to results in the inversion sequences of elements of S_n.

Restricted inversion sequences. Any permutation $\pi' = \pi'_1\pi'_2 \cdots \pi'_n$ in S_n can be coded as an *inversion sequence* $\pi_1\pi_2 \cdots \pi_n$ of length n, where $\pi_j = |\{i \mid \pi'_i < \pi'_{n+1-j},\ n+2-j \leq i \leq n\}|$, for all $j = 1, 2, \ldots, n$. For example, the permutations of S_3, namely 123, 132, 213, 231, 312, and 321, are coded by the following inversion sequences 000, 010, 001, 011, 002, and 012, respectively. Note that an inversion sequence $\pi = \pi_1\pi_2 \cdots \pi_n$ of length n satisfies $0 \leq \pi_i < i$ for all $i = 1, 2, \ldots, n$. We denote the set of all inversion sequences of length n by $\mathcal{I}_n$. Let $\pi = \pi_1\pi_2 \cdots \pi_n \in \mathcal{I}_n$ and let τ be any word contains all the letters $0, 1, \ldots, k$. An *occurrence* of τ in π is a subsequence $1 \leq i_1 < i_2 < \cdots < i_k \leq n$ such that $\pi_{i_1}\pi_{i_2} \cdots \pi_{i_k}$ is order-isomorphic to τ; in such a context τ is usually called a *pattern*. We say that π *avoids* τ, or is τ*-avoiding*, if there is no occurrence of τ in π. We denote the set of all τ-avoiding inversion sequences in $\mathcal{I}_n$ by $\mathcal{I}_n(\tau)$. For an arbitrary finite collection of patterns T, we say that π avoids T if π avoids any $\tau \in T$; we denote the corresponding subset of $\mathcal{I}_n$ by $\mathcal{I}_n(T)$.

Restricted signed inversion sequences. Let us view the elements of B_n as signed permutation $s = s_1s_2 \cdots s_n$ in which each of the symbols $1, 2, \ldots, n$ appears once, possibly barred. Clearly, the cardinality of B_n is $n!2^n$. Let $\theta = \theta_1\theta_2 \cdots \theta_n \in B_n$. We define the barring operation as the one which changes the symbol θ_i to $\overline{\theta_i}$ and $\overline{\theta_i}$ to θ_i. It is thus an involution, that is, $\overline{\overline{\theta_i}} = \theta_i$. Furthermore, we define the absolute value $|a| = |\bar{a}| = a$, for all $a \geq 0$. Any signed permutation $\pi = \pi_1\pi_2 \cdots \pi_n$ in B_n can be coded as an *signed inversion sequence* $\theta = \theta_1\theta_2 \cdots \theta_n$ of length n, where $|\theta_j| = |\{i \mid |\pi_i| < |\pi_{n+1-j}|,\ n+2-j \leq i \leq n\}|$ and θ_j is barred if and only if π_{n+1-j} is barred, for all $j = 1, 2, \ldots, n$. We denote the set of all signed inversion sequences of length n by $\bar{\mathcal{I}}_n$. For example, the singed permutation $\bar{3}1\bar{5}\bar{2}4 \in B_5$ is coded by the following signed inversion sequence $0\bar{0}\bar{2}0\bar{2}$.

Let $\theta = \theta_1\theta_2 \cdots \theta_n \in B_n$ and τ be any word contains all the letters $0, 1, \ldots, k$ possibly barred. We say that θ *contains* τ, if there is a sequence of k indices $1 \leq i_1 < i_2 < \cdots < i_k \leq n$ such that $|\theta_{i_1}||\theta_{i_2}| \cdots |\theta_{i_k}|$ is order-isomorphic to $|\tau_1||\tau_2| \cdots |\tau_k|$ and θ_{i_j} is barred if and only if τ_j is barred, for all $1 \leq j \leq k$. In such a context τ is usually called a *signed pattern*. For example, the signed inversion sequence $0\bar{0}\bar{2}0\bar{2}$ avoids 001 but contains $00\bar{1}$ and the signed inversion sequence $0\bar{0}0\bar{2}\bar{0}$ avoids 000 but contains $0\bar{1}\bar{0}$. We denote the set of all τ-avoiding signed inversion sequences in $\bar{\mathcal{I}}_n$ by $\bar{\mathcal{I}}_n(\tau)$. For an arbitrary finite collection of patterns B, we say that π avoids B if π avoids any $\tau \in B$; we denote the corresponding subset of $\bar{\mathcal{I}}_n$ by $\bar{\mathcal{I}}_n(B)$. We say that two sets of signed patterns B and C are Wilf-equivalent if $|\bar{\mathcal{I}}_n(B)| = |\bar{\mathcal{I}}_n(C)|$, for all $n \geq 0$.

In the symmetric group S_n, for every pattern $\tau \in S_3$, the number of τ-avoiding permutations is given by the nth Catalan number $\frac{1}{n+1}\binom{2n}{n}$, see [2]. Simion [6] showed that there are similar results for the hyperoctahedral group, for every signed pattern $\tau \in B_2$, the number of τ-avoiding signed permutations in B_n is given by $\sum_{j=0}^{n} j!\binom{n}{j}^2$ (For generalization, see [4]).

*Corresponding author (tmansour@univ.haifa.ac.il).

As an extension of these works, several researchers studied restricted inversion sequences. In particular, in [1, 5], explicit formulas and/or generating functions are derived, which count the inversion sequences of a given length that avoid a length-3 pattern. In this paper, we study the number of the signed inversion sequences in $\bar{\mathcal{I}}_n$ that avoid a length-2 signed pattern. In particular, we show the following result:

Theorem 1.1. *For signed inversion sequences, we show that*
(1) the number of Wilf-equivalences among singles of a length-2 signed pattern is 3, and
(2) the number of Wilf-equivalences among pairs of length-2 signed patterns is 30.

In the next section, we extend the use of generating functions from the case of inversion sequences to signed inversion sequences. In Section 3, we prove Theorem 1.1(1), and then in Section 4, we prove Theorem 1.1(2).

2. Generating trees and signed inversion sequences

To establish a meaningful link between generating trees and the problem of avoiding patterns in signed inversion sequences, let us extend the generating trees [7] that discussed in [3] to pattern avoidance on signed inversion sequences. Given a set of patterns B, we define $\bar{\mathcal{I}}(B) = \cup_{n\geq 0}\bar{\mathcal{I}}_n(B)$. We proceed to build a pattern-avoidance tree, denoted as $\mathcal{T}(B)$, for the set $\bar{\mathcal{I}}(B)$. If does not exists an nonempty signed inversion sequence that avoids the set B, then the tree $\mathcal{T}(B)$ has only a root labeled by empty word ϵ. Starting from this initial root that remains at level 0, the nodes at level $n+1$ within the tree $\mathcal{T}(B)$ can be generated from the nodes at level n in a manner that the descendants of $e = e_1 \cdots e_n \in \bar{\mathcal{I}}_n(B)$ are $e' = e_1 \cdots e_n j \in \bar{\mathcal{I}}_{n+1}(B)$, where $j = \bar{n}, \ldots, \bar{1}, \bar{0}, 0, 1, \ldots, n$. Now, we proceed to relabel the vertices of the tree $\mathcal{T}(B)$ as follows. Let $\mathcal{T}(B; e)$ be the subtree comprises the signed inversion sequence e as its root along with its subsequent descendants in $\mathcal{T}(B)$. Let e, e' be any two nodes in $\mathcal{T}(B)$, e is said to be *equivalent* to e', denoted by $e \sim e'$ if and only if $\mathcal{T}(B; e) \cong \mathcal{T}(B; e')$ as plane tree isomorphism. We define the set of all equivalent classes in the quotient set $\mathcal{T}(B)/\sim$ by $\mathcal{E}(B)$. We denote each equivalence class in $\mathcal{E}(B)$ by the label of the singular node found on the tree $\mathcal{T}(B)$ as the first node (from top to bottom, left to right). Let $\mathcal{T}[B]$ be the identical tree $\mathcal{T}(B)$, where each node of $\mathcal{T}(B)$ is replaced with its corresponding equivalence class label.

Example 2.1. *Let* $B = \{00, \bar{0}\bar{0}, 01, 0\bar{1}\}$. *The left side of Figure 2.1 presents the first levels of the generating tree* $\mathcal{T}(B)$. *The generating tree* $\mathcal{T}[B]$ *is given by*

$$
\begin{aligned}
\epsilon &\rightsquigarrow \bar{0}, 0,\\
0 &\rightsquigarrow 0\bar{0},\\
\bar{0} &\rightsquigarrow a_1, 0\bar{0}, \bar{0}1,\\
\bar{0}1 &\rightsquigarrow 0\bar{0}, 0,\\
a_m &\rightsquigarrow a_{m+1}, 0\bar{0}, 0, b_2, \ldots, b_m, c_m,\\
b_m &\rightsquigarrow 0\bar{0}, 0, b_2, \ldots, b_{m-1},\\
c_m &\rightsquigarrow b_{m+1}, 0\bar{0}, 0, b_2, \ldots, b_m,
\end{aligned}
$$

where $a_m = \bar{0}\bar{1}\cdots\bar{m}$, $b_m = \bar{0}\bar{1}\cdots\bar{m}m$, *and* $c_m = \bar{0}\bar{1}\cdots\bar{m}(m+1)$. *For instance, the right side of Figure 2.1 presents the first levels of the generating tree* $\mathcal{T}[\{00, \bar{0}\bar{0}, 01, 0\bar{1}\}]$.

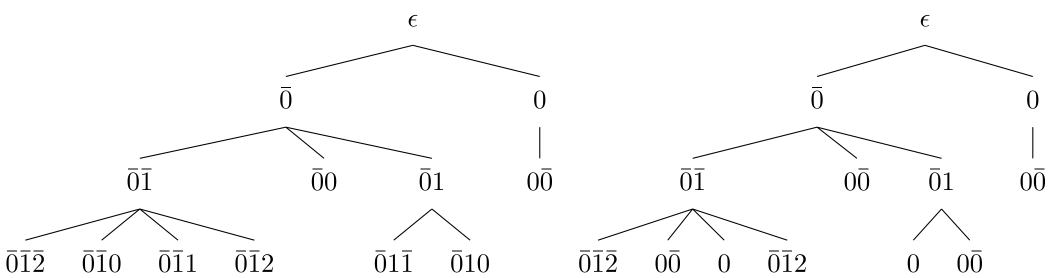

Figure 2.1: The generating trees $\mathcal{T}(\{00, \bar{0}\bar{0}, 01, 0\bar{1}\})$ and $\mathcal{T}[\{00, \bar{0}\bar{0}, 01, 0\bar{1}\}]$.

Now, let us prove the rules of $\mathcal{T}[B]$*, where* $B=\{00,\bar{0}\bar{0},01,0\bar{1}\}$*:*

- *the children of* ϵ *are* $\bar{0},0$*. Thus,* $\epsilon \rightsquigarrow \bar{0},0$*.*
- *the only child of* 0 *is* $0\bar{0}$*, so* $0 \rightsquigarrow 0\bar{0}$*. Note that there are no children for* $0\bar{0}$*.*
- *the children of* $\bar{0}$ *are* $\bar{0}\bar{1}$*,* $\bar{0}0$*,* $\bar{0}1$*. Note that* $\mathcal{T}(B;\bar{0}0)\cong\mathcal{T}(B;0\bar{0})$*. So,* $\bar{0}\rightsquigarrow a_1,0\bar{0},\bar{0}1$*.*
- *the children of* $\bar{0}1$ *are* $\bar{0}1\bar{1}$ *and* $\bar{0}10$*. Note that* $\mathcal{T}(B;\bar{0}1\bar{1})\cong\mathcal{T}(B;0)$ *(the only child of* $\bar{0}1\bar{1}$ *is* $\bar{0}1\bar{1}0$ *and there are no children for* $\bar{0}1\bar{1}0$*) and* $\mathcal{T}(B;\bar{0}10)\cong\mathcal{T}(B;\bar{0}0)\cong\mathcal{T}(B;0\bar{0})$ *(by removing the second letter from any signed inversion sequence of the form* $\bar{0}10\pi'$*). Thus,* $\bar{0}1\rightsquigarrow 0,0\bar{0}$*.*
- *the children of* a_m *are* $a_{m+1}=a_m\overline{m+1}$*,* a_mj *with* $j=0\dots,m-1$*,* $b_m=a_mm$*, and* $c_m=a_m(m+1)$*. Note that* $\mathcal{T}(B;a_m0)\cong(\mathcal{T}(B;0\bar{0})$ *(for the signed inversion sequence* a_m0 *there are no children),* $\mathcal{T}(B;a_m1)\cong\mathcal{T}(B;\bar{0}1\bar{1})\cong\mathcal{T}(B;0)$ *(see the previous item), and* $\mathcal{T}(B;a_mj)\cong\mathcal{T}(B;b_j)$ *(by removing the any letter* k *from a signed inversion sequence* $a_mj\pi'$ *such that* $|k|>j$*) for* $j=2,3,\dots,m-1$*. Thus,* $a_m\rightsquigarrow a_{m+1},0\bar{0},0,b_2,\dots,b_m,c_m$*;*
- *Similarly, we have that* $b_m\rightsquigarrow 0\bar{0},0,b_2,\dots,b_{m-1}$ *and* $c_m\rightsquigarrow b_{m+1},0\bar{0},0,b_2,\dots,b_m$*.*

After we guessed and proved (if possible) the rules of the generating tree $\mathcal{T}[B]$, we translate these rules into a system of equations and we solve for $F_B(x)=\sum_{n\geq 0}|\bar{\mathcal{I}}_n(B)|x^n$. Note that the rule $e\rightsquigarrow v^{(1)},\dots,v^{(s)}$ can be translated to

$$I_e(x)=1+x\sum_{j=1}^{s}I_{v^{(j)}}(x),$$

where $I_w(x)=\sum_{n\geq 0}(\#\text{the nodes at level } n \text{ in } \mathcal{T}(B;w))x^n$ is the generating function for the number of nodes at level $n\geq 0$ in the subtree of $\mathcal{T}(B;w)$, where its root stays at level 0. Clearly, $F_B(x)=I_\epsilon(x)$.

Example 2.2. *As continuation of Example 2.1, we have*

$$\begin{aligned}
I_\epsilon(x)&=1+xI_{\bar{0}}(x)+xI_0(x),\\
I_0(x)&=1+xI_{0\bar{0}}(x),\\
I_{\bar{0}}(x)&=1+xA_1(x)+xI_{0\bar{0}}(x)+xI_{\bar{0}1}(x),\\
I_{\bar{0}1}(x)&=1+xI_{0\bar{0}}(x)+xI_0(x),\\
A_m(x)&=1+xA_{m+1}(x)+xI_{0\bar{0}}+xI_0(x)+x\sum_{j=2}^{m}B_j(x)+xC_m(x),\\
B_m(x)&=1+xI_{0\bar{0}}(x)+xI_0(x)+x\sum_{j=2}^{m-1}B_j(x),\\
C_m(x)&=1+xI_{0\bar{0}}(x)+xI_0(x)+x\sum_{j=2}^{m+1}B_j(x),
\end{aligned}$$

where $A_m(x)=I_{a_m}(x)$*,* $B_m(x)=I_{b_m}(x)$*, and* $C_m(x)=I_{c_m}(x)$*.*

Note that $I_{0\bar{0}}(x)=1$ *and* $I_0(x)=1+x$*. So, by induction on* m*, we have that* $B_m(x)=(1+x)^m$*. This implies that* $C_m(x)=(1+x)^{m+2}$*, and*

$$A_m(x)=xA_{m+1}(x)+(1+x)^{m+1}+x(1+x)^{m+2}.$$

By iterating this equation infinity number of times (here we assumed that $|x|<1$*), we obtain*

$$\begin{aligned}
A_1(x)&=(1+x)^2+x(1+x)^3+xA_2(x)\\
&=(1+x)^2(1+x(1+x))+x(1+x)^3(1+x(1+x))+x^2A_3(x)\\
&=(1+x)^2(1+x(1+x)+x^2(1+x)^2)+x(1+x)^3(1+x(1+x)+x^2(1+x)^2)+x^3A_4(x)\\
&=\cdots=(1+x)^2\sum_{j\geq 0}x^j(1+x)^j+x(1+x)^3\sum_{j\geq 0}x^j(1+x)^j\\
&=\frac{(1+x)^2(1+x+x^2)}{1-x-x^2}.
\end{aligned}$$

Hence, by solving

$$I_\epsilon(x) = 1 + xI_{\bar{0}}(x) + xI_0(x),$$
$$I_0(x) = 1 + xI_{0\bar{0}}(x),$$
$$I_{\bar{0}}(x) = 1 + xA_1(x) + xI_{0\bar{0}}(x) + xI_{\bar{0}1}(x),$$
$$I_{\bar{0}1}(x) = 1 + xI_{0\bar{0}}(x) + xI_0(x),$$

for $I_\epsilon(x)$, we obtain

$$F_B(x) = I_\epsilon(x) = \frac{1+x+x^2}{1-x-x^2}.$$

3. Avoiding a length-2 signed pattern

In this section, we deal with avoiding a length-2 signed pattern. To shorten the notation, we define S^m to be either the word $SS\cdots S$ with m letters or the sequence $S, S, \ldots, S$ with m terms, for any sequence S and a nonnegative integer m. By finding the number of signed inversion sequences of length n that avoid a length-2 signed pattern, for all $n = 1, 2, \ldots, 7$, we can guess that there are three Wilf classes, see Table 1.

τ	$\{\lvert\bar{\mathcal{I}}_n(\tau)\rvert\}_{n=1,2,\ldots,7}$
$00, 0\bar{0}, \bar{0}0, \bar{0}\bar{0}$	2, 7, 36, 245, 2076, 21059, 248836
$01, 0\bar{1}, \bar{0}1, \bar{0}\bar{1}$	2, 7, 33, 193, 1342, 10796, 98552
$10, 1\bar{0}, \bar{1}0, \bar{1}\bar{0}$	2, 8, 46, 338, 2992, 30800, 360110

Table 1: Number signed inversion sequences in $\bar{\mathcal{I}}_n(\tau)$, where $n = 1, 2, \ldots, 7$ and τ is any length-2 signed pattern.

Theorem 3.1. *We have*

(1) $00 \sim 0\bar{0} \sim \bar{0}0 \sim \bar{0}\bar{0}$*,*

(2) $01 \sim 0\bar{1} \sim \bar{0}1 \sim \bar{0}\bar{1}$*,*

(3) $10 \sim 1\bar{0} \sim \bar{1}0 \sim \bar{1}\bar{0}$*.*

Proof. (2). By symmetric operation baring, we see that $01 \sim \bar{0}\bar{1}$ and $0\bar{1} \sim \bar{0}1$. Hence, it is enough to show that $01 \sim 0\bar{1}$. Note that the generating trees $\mathcal{T}[01]$ and $\mathcal{T}[0\bar{1}]$ are given by a root ϵ and the following rules

$$\epsilon \rightsquigarrow a_1, b_{0,0},$$
$$a_m \rightsquigarrow (a_{m+1})^{m+1}, b_{m,0}, b_{m,1}, \ldots, b_{m,m},$$
$$b_{m,j} \rightsquigarrow (b_{m+1,j})^{m+2}, b_{m+1,0}, b_{m+1,1}, \ldots, b_{m+1,j},$$

where $a_m = \bar{0}^m$ and $b_{m,j} = a_m j$. To prove these are the rules of $\mathcal{T}[01]$ (similarly, $\mathcal{T}[0\bar{1}]$), we have to study the children of a_m and $b_{m,j}$. Note that the children of ϵ are $a_1 = \bar{0}$ and $b_{0,0} = 0$, so the rule $\epsilon \rightsquigarrow a_1, b_{0,0}$ holds. Also, the children of a_m are exactly $a_m\bar{j}$ with $j = 0, 1, \ldots, m$ and $b_{m,0}, b_{m,1}, \ldots, b_{m,m}$, where $\mathcal{T}(01, a_m\bar{j}) \cong \mathcal{T}(01, a_{m+1})$. Hence, the rule for a_m is given by $a_m \rightsquigarrow (a_{m+1})^{m+1}, b_{m,0}, b_{m,1}, \ldots, b_{m,m}$. Moreover, the children of $b_{m,j}$ are $b_{m,j}\bar{i}$ with $i = 0, 1, \ldots, m+1$, and $b_{m,j}i$ with $i = 0, 1, \ldots, j$, where $\mathcal{T}(01, bm, j\bar{i}) \cong \mathcal{T}(01, b_{m+1,j})$ and $\mathcal{T}(01, b_{m,j}i) \cong \mathcal{T}(01, b_{m+1,i})$. Hence, the rule for a_m is given by $b_{m,j} \rightsquigarrow (b_{m+1,j})^{m+2}, b_{m+1,0}, b_{m+1,1}, \ldots, b_{m+1,j}$.

(1). By symmetric operation baring, we see that $00 \sim \bar{0}\bar{0}$ and $0\bar{0} \sim \bar{0}0$. Hence, it is enough to show that $00 \sim 0\bar{0}$. Note that the generating tree $\mathcal{T}[00]$ is given by a root ϵ and the following rules:

$$\epsilon \rightsquigarrow a_{0;}, b_{0;},$$
$$a_{m;i_1\cdots i_s} \rightsquigarrow (a_{m+1;i_1\cdots i_s})^{m+s+2}, (b_{m;ji_1\cdots i_s})_{j=1}^{i_1-1}, (b_{m;i_1ji_2\cdots i_s})_{j=i_1+1}^{i_2-1}, \ldots, (b_{m;i_1\cdots i_sj})_{j=i_s+1}^{m+s+1},$$
$$b_{m;i_1\cdots i_s} \rightsquigarrow (b_{m+1;i_1\cdots i_s})^{m+s+2}, (b_{m;ji_1\cdots i_s})_{j=0}^{i_1-1}, (b_{m;i_1ji_2\cdots i_s})_{j=i_1+1}^{i_2-1}, \ldots, (b_{m;i_1\cdots i_sj})_{j=i_s+1}^{m+s+1},$$

where $a_{m;i_1\cdots i_s} = 0\bar{0}^m i_1 \cdots i_s$ and $b_{m,j} = \bar{0}^m i_1 \cdots i_s$.

Also, the generating tree $\mathcal{T}[0\bar{0}]$ is given by a root ϵ and the following rules:

$$\begin{aligned}\epsilon &\rightsquigarrow c_{0;}, d_{0;},\\ c_{m;i_1\cdots i_s} &\rightsquigarrow (c_{m+1;i_1\cdots i_s})^{m+s+2}, (d_{m;ji_1\cdots i_s})_{j=1}^{i_1-1}, (d_{m;i_1ji_2\cdots i_s})_{j=i_1+1}^{i_2-1}, \ldots, (d_{m;i_1\cdots i_sj})_{j=i_s+1}^{m+s+1},\\ d_{m;i_1\cdots i_s} &\rightsquigarrow (d_{m+1;i_1\cdots i_s})^{m+s+2}, (d_{m;ji_1\cdots i_s})_{j=0}^{i_1-1}, (d_{m;i_1ji_2\cdots i_s})_{j=i_1+1}^{i_2-1}, \ldots, (d_{m;i_1\cdots i_sj})_{j=i_s+1}^{m+s+1},\end{aligned}$$

where $c_{m;i_1\cdots i_s} = 0\bar{1}^m i_1\cdots i_s$ and $d_{m,j} = \bar{1}^m i_1\cdots i_s$. We leave to the reader that the rules of the generating trees $\mathcal{T}[00]$ and $\mathcal{T}[0\bar{0}]$ are holding (similar to the proofs of the Case (2)). Hence, $\mathcal{T}[00] \cong \mathcal{T}[0\bar{0}]$ by mapping $a_{m;i_1\cdots i_s}$ to $c_{m;i_1\cdots i_s}$ and $b_{m;i_1\cdots i_s}$ to $d_{m;i_1\cdots i_s}$. Hence, $00 \sim 0\bar{0}$.

(3). By symmetric operation baring, we see that $10 \sim \bar{1}\bar{0}$ and $1\bar{0} \sim \bar{1}0$. Hence, it is enough to show that $10 \sim 1\bar{0}$. Note that the generating trees $\mathcal{T}[10]$ and $\mathcal{T}[1\bar{0}]$ are given by a root ϵ and the following rules

$$\begin{aligned}\epsilon &\rightsquigarrow a_1, a_1,\\ a_m &\rightsquigarrow (a_{m+1})^{m+2}, b_{m,1}, b_{m,2}, \ldots, b_{m,m},\\ b_{m,j} &\rightsquigarrow (b_{m+1,j})^{m+3}, b_{m+1,j+1}, b_{m+1,j+2}, \ldots, b_{m+1,m+1},\end{aligned}$$

where $a_m = \bar{0}^m$ and $b_{m,j} = a_m j$. We leave to the reader that the rules of the generating trees $\mathcal{T}[10]$ and $\mathcal{T}[1\bar{0}]$ are holding (similar to the proofs of the Case (2)). Hence, $\mathcal{T}[00] \cong \mathcal{T}[0\bar{0}]$, which implies that $00 \sim 0\bar{0}$ □

4. Avoiding a pair of length-2 signed patterns

In this section, we deal with avoiding a pair of length-2 signed patterns. By finding the number of signed inversion sequences of length n that avoid a length-2 signed pattern, for all $n = 1, 2, \ldots, 7$, we can guess that there are 30 Wilf classes, see Table 2. Note that by baring operation, we see that $|\bar{\mathcal{I}}_n(B)| = |\bar{\mathcal{I}}_n(\bar{B})|$, for all $n \geq 0$. So, in Table 2, we do include the baring of a set B if B is included.

Based on Table 2, in order to find the number of Wilf classes when signed inversion sequences of length n avoid a pair of length-2 signed pattern, we have to consider the Classes $6, 11, 12, 19, 26$ in Table 2. In the next 5 propositions, we prove that each class of these classes create exactly one Wilf class.

Proposition 4.1. *We have* $\{00, \bar{0}\bar{0}\} \sim \{01, \bar{0}1\}$.

Proof. Let $B = \{01, \bar{0}1\}$. Clearly, the children of the root ϵ of $\mathcal{T}(B)$ are 0 and $\bar{0}$. But with baring we see that $\mathcal{T}(B; 0) \cong \mathcal{T}(B; \bar{0})$. Thus, $\epsilon \rightsquigarrow 0, 0$. The children of 0^m are $0^m\bar{m}, \ldots, 0^m\bar{0}, 0^{m+1}$. Clearly, $\mathcal{T}(B; 0^m\bar{j}) \cong \mathcal{T}(B; 0^{m+1})$. Thus, $0^m \rightsquigarrow (0^{m+1})^{m+2}$. Hence, the generating tree $\mathcal{T}[B]$ is given by

$$\begin{aligned}\epsilon &\rightsquigarrow 0, 0\\ 0^m &\rightsquigarrow (0^{m+1})^{m+2}.\end{aligned}$$

Let $B = \{00, \bar{0}\bar{0}\}$. Clearly, the children of the root ϵ of $\mathcal{T}(B)$ are 0 and $\bar{0}$. But with baring we see that $\mathcal{T}(B; 0) \cong \mathcal{T}(B; \bar{0})$. Thus, $\epsilon \rightsquigarrow 0, 0$. The children of $01\cdots(m-1)$ are $01\cdots(m-1)\bar{m}, \ldots, 01\cdots(m-1)\bar{0}, 01\cdots(m-1)m$. By exchanging the letters $\bar{j}$ and m, we have $\mathcal{T}(B; 01\cdots(m-1)\bar{j}) \cong \mathcal{T}(B; 01\cdots m)$. Thus, $01\cdots(m-1) \rightsquigarrow (01\cdots m)^{m+2}$. Hence, the generating tree $\mathcal{T}[B]$ is given by

$$\begin{aligned}\epsilon &\rightsquigarrow 0, 0\\ 01\cdots(m-1) &\rightsquigarrow (01\cdots m)^{m+2}.\end{aligned}$$

Hence, by mapping the node 0^m of $\mathcal{T}[01, \bar{0}1]$ to the node $01\cdots(m-1)$ of $\mathcal{T}[00, \bar{0}\bar{0}]$, we see that $\mathcal{T}[00, \bar{0}\bar{0}] \cong \mathcal{T}[01, \bar{0}1]$, which implies $\{00, \bar{0}\bar{0}\} \sim \{01, \bar{0}1\}$. □

As a corollary from Proposition 4.1, we see that the number of signed inversion sequences of length n that avoid $\{01, \bar{0}1\}$ is given by $(n+1)!$, for all $n \geq 0$.

1	$\{01, \bar{0}\bar{1}\}$	2, 6, 18, 52, 146, 402, 1092
2	$\{0\bar{1}, \bar{0}1\}$	2, 6, 20, 76, 344, 1888, 12416
3	$\{00, \bar{0}\bar{1}\}$	2, 6, 21, 81, 341, 1558, 7679
4	$\{00, \bar{0}1\}$	2, 6, 22, 98, 524, 3298, 23960
5	$\{01, 0\bar{1}\}$	2, 6, 22, 98, 526, 3338, 24526
6	$\{00, \bar{0}\bar{0}\}$,$\{01, \bar{0}1\}$	2, 6, 24, 120, 720, 5040, 40320
7	$\{00, 0\bar{1}\}$	2, 6, 24, 120, 722, 5092, 41252
8	$\{0\bar{0}, \bar{0}1\}$	2, 6, 24, 121, 736, 5239, 42693
9	$\{0\bar{0}, \bar{0}\bar{1}\}$	2, 6, 24, 121, 737, 5263, 43107
10	$\{0\bar{0}, 01\}$	2, 6, 25, 132, 842, 6288, 53766
11	$\{00, 01\}$,$\{0\bar{0}, 0\bar{1}\}$	2, 6, 25, 133, 859, 6516, 56710
12	$\{00, 0\bar{0}\}$,$\{00, \bar{0}0\}$,$\{0\bar{0}, \bar{0}0\}$	2, 6, 26, 150, 1082, 9366, 94586
13	$\{01, \bar{1}\bar{0}\}$	2, 7, 31, 155, 834, 4717, 27675
14	$\{0\bar{1}, 10\}$	2, 7, 31, 159, 917, 5896, 42231
15	$\{01, \bar{1}0\}$	2, 7, 31, 162, 979, 6766, 52924
16	$\{0\bar{1}, 1\bar{0}\}$	2, 7, 31, 162, 982, 6836, 54060
17	$\{0\bar{1}, \bar{1}\bar{0}\}$	2, 7, 32, 175, 1106, 7943, 64128
18	$\{01, 1\bar{0}\}$	2, 7, 32, 177, 1143, 8430, 69920
19	$\{01, 10\}$,$\{0\bar{1}, \bar{1}0\}$	2, 7, 32, 178, 1164, 8748, 74304
20	$\{00, \bar{1}\bar{0}\}$	2, 7, 34, 204, 1429, 11314, 99153
21	$\{0\bar{0}, \bar{1}0\}$	2, 7, 34, 207, 1500, 12542, 118506
22	$\{00, 1\bar{0}\}$	2, 7, 34, 208, 1521, 12874, 123410
23	$\{0\bar{0}, 10\}$	2, 7, 34, 209, 1546, 13327, 130922
24	$\{00, \bar{1}0\}$	2, 7, 35, 224, 1735, 15716, 162618
25	$\{0\bar{0}, \bar{1}\bar{0}\}$	2, 7, 35, 225, 1757, 16085, 168484
26	$\{00, 10\}$,$\{0\bar{0}, 1\bar{0}\}$	2, 7, 35, 226, 1780, 16489, 175191
27	$\{10, \bar{1}\bar{0}\}$	2, 8, 44, 292, 2192, 17948, 156740
28	$\{1\bar{0}, \bar{1}0\}$	2, 8, 44, 292, 2204, 18332, 164924
29	$\{10, 1\bar{0}\}$	2, 8, 44, 296, 2312, 20384, 199376
30	$\{10, \bar{1}0\}$	2, 8, 44, 300, 2420, 22460, 235260

Table 2: Number of signed inversion sequences in $\bar{\mathcal{I}}_n(B)$, where $n = 1, 2, \ldots, 7$ and B is any pair of length-2 signed patterns.

Similar to the proof of Proposition 4.1, we obtain the next result.

Proposition 4.2. *The generating trees* $\mathcal{T}[00, 01]$ *and* $\mathcal{T}[0\bar{0}, 0\bar{1}]$ *have a root* ϵ *and satisfy*

$$\begin{aligned} &\epsilon \rightsquigarrow a_1, b_{0,0}, \\ &a_m \rightsquigarrow (a_{m+1})^{m+1}, b_{m,0}, \ldots, b_{m,m}, \\ &b_{m,j} \rightsquigarrow (b_{m+1,j})^{m+2}, b_{m+1,0}, \ldots, b_{m+1,j-1}, \end{aligned}$$

where $a_m = \bar{0}^m$ *and* $b_{m,j} = a_m j$. *Thus,* $\mathcal{T}[00, 01] \cong \mathcal{T}[0\bar{0}, 0\bar{1}]$, *which implies that* $\{00, 01\} \sim \{0\bar{0}, 0\bar{1}\}$.

Proposition 4.3. *We have* $\{00, 0\bar{0}\} \sim \{00, \bar{0}0\} \sim \{0\bar{0}, \bar{0}0\}$.

Proof. We proceed by showing there are bijections $\alpha : \bar{\mathcal{I}}_n(\{00, 0\bar{0}\}) \leftrightarrow \bar{\mathcal{I}}_n(\{00, \bar{0}0\})$ and $\beta : \bar{\mathcal{I}}_n(\{00, \bar{0}0\}) \leftrightarrow \bar{\mathcal{I}}_n(\{0\bar{0}, \bar{0}0\})$. Let $\pi = \pi_1 \cdots \pi_n \in \bar{\mathcal{I}}_n$ be any signed inversion sequence, we say $\pi_{i_1} \cdots \pi_{i_s}$ is a (s, d)-level if exists $s \geq 1$ and exist $i_1 < \cdots < i_s$ such that $d = |\pi_{i_1}| = \cdots = |\pi_{i_s}| \neq |\pi_j|$, for all $j \notin \{i_1, \ldots, i_s\}$.

Now, let us define α. Let $\pi \in \bar{\mathcal{I}}_n(\{00, 0\bar{0}\})$. Clearly, the (s, d)-level of π forms a sequence of type either $\bar{d}\bar{d} \cdots \bar{d}d$ or $\bar{d}\bar{d} \cdots \bar{d}$. We define $\alpha(\pi)$ to be π after changing the (s, d)-level $\bar{d}\bar{d} \cdots \bar{d}d$ of π to $d\bar{d} \cdots \bar{d}\bar{d}$, for all $d = 0, 1, \ldots, n$. Thus, $\pi \in \bar{\mathcal{I}}_n(\{00, 0\bar{0}\})$ if and only if $\alpha(\pi) \in \bar{\mathcal{I}}_n(\{00, \bar{0}0\})$. Hence, $\{00, 0\bar{0}\} \sim \{00, \bar{0}0\}$.

Now, let us define β. Let $\pi \in \bar{\mathcal{I}}_n(\{00, \bar{0}0\})$. Clearly, the (s, d)-level of π forms a sequence of type either $d\bar{d} \cdots \bar{d}$ or $\bar{d}\bar{d} \cdots \bar{d}$. We define $\beta(\pi)$ to be π after changing the (s, d)-level $d\bar{d} \cdots \bar{d}$ of π with $s \geq 1$ to $dd \cdots d$, for all $d = 0, 1, \ldots, n$. Thus, $\pi \in \bar{\mathcal{I}}_n(\{00, 0\bar{0}\})$ if and only if $\beta(\pi) \in \bar{\mathcal{I}}_n(\{0\bar{0}, \bar{0}0\})$. Hence, $\{00, \bar{0}0\} \sim \{0\bar{0}, \bar{0}0\}$. □

Proposition 4.4. *We have* $\{01, 10\} \sim \{0\bar{1}, \bar{1}0\}$.

Proof. We proceed by showing there is a bijection $\alpha : \bar{\mathcal{I}}_n(\{01, 10\}) \leftrightarrow \bar{\mathcal{I}}_n(\{0\bar{1}, \bar{1}0\})$. Let $\pi = \pi_1 \cdots \pi_n \in \bar{\mathcal{I}}_n$, we say that π has a minimal level k if there exist i such that $\pi_i = k$ and $\pi_j = \bar{s}$ for any $0 \leq s < k$ and $1 \leq j \leq n$. For example, $\pi = \bar{0}\bar{1}\bar{0}\bar{1}\bar{2}32\bar{3}\bar{1}\bar{0}$ has a minimal level 3. If k does not exist, then any letter in π is barred.

Now, let $\pi = \pi_1 \cdots \pi_n \in \bar{\mathcal{I}}_n(\{01, 10\})$. If π has a minimal level k, then any letter π_j is barred whenever $|\pi_j| > k$. Define $\alpha(\pi) = \pi'_1 \cdots \pi'_n$, where $\pi'_j = \pi_j$ if $|\pi_j| \leq k$ and $\pi'_j = \bar{\pi}_j$ if $|\pi_j| > k$. Thus, any letter π'_j of $\alpha(\pi)$ is not barred (respectively, barred) whenever $|\pi'_j| > k$ (respectively, $|\pi'_j| < k$). Thus, π avoids $\{01, 10\}$ if and only if $\alpha(\pi)$ avoids $\{0\bar{1}, \bar{1}0\}$. Hence, $\{01, 10\} \sim \{0\bar{1}, \bar{1}0\}$. □

To count the class $B = \{01, 10\}$, we find the generating tree $\mathcal{T}[B]$. Clearly, the children of the root ϵ of $\mathcal{T}[B]$ are 0 and $\bar{0}$. Thus, $\epsilon \rightsquigarrow 0, \bar{0}$. The children of $\bar{0}^m$ are $\bar{0}^m\bar{m}, \ldots, \bar{0}^m\bar{0}, \bar{0}^m 0, \ldots, \bar{0}^m m$. Clearly, $\mathcal{T}(B; \bar{0}^m\bar{j}) \cong \mathcal{T}(B; \bar{0}^{m+1})$. Thus, $\bar{0}^m \rightsquigarrow (\bar{0}^{m+1})^{m+1}, \bar{0}^m 0, \ldots, \bar{0}^m m$. Moreover, The children of $\bar{0}^m j$ are $\bar{0}^m j\overline{m+1}, \ldots, \bar{0}^m j\bar{0}, \bar{0}^m jj$. Clearly, $\mathcal{T}(B; \bar{0}^m j\bar{j}) \cong \mathcal{T}(B; \bar{0}^{m+1} j)$ and $\mathcal{T}(B; \bar{0}^m jj) \cong \mathcal{T}(B; \bar{0}^{m+1} j)$. Thus, $\bar{0}^m j \rightsquigarrow (\bar{0}^{m+1} j)^{m+3}$. Hence, the generating tree $\mathcal{T}[B]$ is given by

$$\begin{aligned}
&\epsilon \rightsquigarrow 0, \bar{0}\\
&\bar{0}^m \rightsquigarrow (\bar{0}^{m+1})^{m+1}, \bar{0}^m 0, \ldots, \bar{0}^m m,\\
&\bar{0}^m j \rightsquigarrow (\bar{0}^{m+1} j)^{m+3}.
\end{aligned}$$

Let $B_{m,j}(x) = I_{\bar{0}^m j}(x)$ and $B_m(x) = I_{\bar{0}^m}(x)$. Then

$$B_{m,j}(x) = 1 + (m+3)xB_{m+1,j}(x).$$

By iterating, we have (from now, we assume that $|x| < 1$)

$$\begin{aligned}
B_{m,j}(x) &= 1 + (m+3)xB_{m+1;j}(x)\\
&= 1 + (m+3)x + (m+3)(m+4)x^2 B_{m+2;j}(x)\\
&= \cdots = 1 + (m+3)x + (m+3)(m+4)x^2 + (m+3)(m+4)(m+5)x^3 + \cdots\\
&= \sum_{i\geq 0} \frac{(m+2+i)!}{(m+2)!} x^i.
\end{aligned}$$

Moreover, $B_m(x) = 1 + (m+1)xB_{m+1}(x) + x\sum_{j=0}^{m} B_{m,j}(x)$, which implies

$$B_m(x) = 1 + (m+1)xB_{m+1}(x) + (m+1)x \sum_{i\geq 0} \frac{(m+2+i)!}{(m+2)!} x^i.$$

By induction on m,

$$\begin{aligned}
B_m(x) &= \sum_{i\geq 0} \frac{(m+i)!}{m!} x^i \left(1 + \sum_{i'\geq 0} \frac{(m+i+1)(m+2+i+i')!}{(m+2+i)!} x^{i'+1}\right)\\
&= \sum_{i\geq 0} \frac{(m+i)!}{m!} x^i + \sum_{i\geq 0}\sum_{i'\geq 0} \frac{(m+i+1)(m+2+i+i')!(m+i)!}{(m+2+i)!m!} x^{i'+i+1}.
\end{aligned}$$

Note that $I_\epsilon(x) = 1 + xI_0(x) + xI_{\bar{0}}(x)$, which is equivalent to $I_\epsilon(x) = 1 + xB_{0,0}(x) + xB_1(x)$. Thus,

$$I_\epsilon(x) = 1 + x\sum_{i\geq 0} \frac{(i+2)!}{2} x^i + x\sum_{i\geq 0} (i+1)! x^i + x\sum_{i\geq 0}\sum_{i'\geq 0} \frac{(i+i'+3)!}{i+3} x^{i'+i+1}.$$

Hence, the number of signed inversion sequences of length n that avoid B is given by

$$n! + (n+1)!(H_{n+1} - 1),$$

where $H_n = \sum_{k=1}^{n} \frac{1}{k}$ is the nth Harmonic number.

Similar to the proof of Proposition 4.1, we obtain the following result:

Proposition 4.5. *The generating trees $\mathcal{T}[00, 10]$ and $\mathcal{T}[0\bar{0}, 1\bar{0}]$ have a root ϵ and satisfy*

$$\begin{aligned}
&\epsilon \rightsquigarrow a_1, b_{0,0},\\
&a_m \rightsquigarrow (a_{m+1})^{m+1}, b_{m,0}, \ldots, b_{m,m},\\
&b_{m,j} \rightsquigarrow (b_{m+1,j})^{m+2}, b_{m+1,j+1}, \ldots, b_{m+1,m+1},
\end{aligned}$$

where $a_m = \bar{0}^m$ and $b_{m,j} = a_m j$. Thus, $\mathcal{T}[00, 10] \cong \mathcal{T}[0\bar{0}, 1\bar{0}]$, which implies that $\{00, 10\} \sim \{0\bar{0}, 1\bar{0}\}$.

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