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Research Article

The harmonic index and some Hamiltonian properties of graphs

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Abstract

Let $G = (V, E)$ be a graph. The harmonic index of G is defined as $\sum_{uv \in E} \frac{2}{d(u)+d(v)}$, where $d(u)$ and $d(v)$ denote the degrees of vertices u and v in G, respectively. In this paper, conditions involving the harmonic index for some Hamiltonian properties of a graph are presented. An upper bound for the harmonic index of a graph is also presented.

Keywords: harmonic index; Hamiltonian graph; traceable graph.

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1. Introduction

We consider only finite undirected graphs without loops or multiple edges. Notations and terminologies not defined here follow those in [1]. Let $G = (V(G), E(G))$ be a graph with n vertices and e edges. The degree of a vertex v is denoted by $d_G(v)$ (or simply, $d(v)$). We use δ and Δ to denote the minimum degree and maximum degree of G, respectively. A set of vertices in a graph G is independent if the vertices in the set are pairwise nonadjacent. A maximum independent set in a graph G is an independent set of the largest possible size. The independence number, denoted by $\beta(G)$, of a graph G is the cardinality of a maximum independent set in G. For disjoint vertex subsets X and Y of $V(G)$, we use $E(X, Y)$ to denote the set of all the edges in $E(G)$ such that one end vertex of each edge is in X and the other end vertex of the edge is in Y. Particularly, $E(X, Y) := \{ xy \in E(G) : x \in X, y \in Y \}$. A cycle C in a graph G is called a Hamiltonian cycle of G if C contains all the vertices of G. A graph G is called Hamiltonian if G has a Hamiltonian cycle. A path P in a graph G is called a Hamiltonian path if P contains all the vertices of G. A graph G is called traceable if G has a Hamiltonian path.

The first Zagreb index was introduced by Gutman and Trinajstić in [4]. For a graph G, the first Zagreb index is defined as $\sum_{v \in V(G)} d^2(v) = \sum_{uv \in E(G)} (d(u)+d(v))$. Zhou and Trinajstić in [7] introduced the concept of the general sum-connectivity index of a graph. The general sum-connectivity index, denoted by $\chi_\alpha(G)$, of a graph G is defined as $\sum_{uv \in E(G)} (d(u)+d(v))^\alpha$, where α is a real number such that $\alpha \neq 0$. Obviously, $\chi_1(G)$ is the same as the first Zagreb index of a graph G. Also, $2\chi_{-1}(G)$ is known as the harmonic index, denoted by $H(G)$, of a graph G. In this paper, we use the harmonic index of a graph to obtain sufficient conditions for Hamiltonian and traceable graphs. The main results are as follows.

Theorem 1.1. *Let G be a k-connected graph with n vertices and e edges, where $k \geq 2$ and $n \geq 3$. If*

$$H(G) \geq \frac{(\delta + \Delta)^2}{2M\delta\Delta} e^2,$$

then G is Hamiltonian, where

$$M = (k+1)\delta^2 + \frac{e^2}{n-(k+1)}.$$

Theorem 1.2. *Let G be a k-connected with n vertices and e edges, where $k \geq 1$ and $n \geq 9$. If*

$$H(G) \geq \frac{(\delta + \Delta)^2}{2N\delta\Delta} e^2,$$

then G is traceable, where

$$N = (k+2)\delta^2 + \frac{e^2}{n-(k+2)}.$$

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2. Lemmas

In order to prove Theorems 1.1 and 1.2, we need the following known results:

Lemma 2.1 (see [2]). *Let G be a k-connected graph of order $n \geq 3$. If $\beta \leq k$, then G is Hamiltonian.*

Lemma 2.2 (see [2]). *Let G be a k-connected graph of order n. If $\beta \leq k+1$, then G is traceable.*

Lemma 2.3 (see [5]). *Let G be a balanced bipartite graph of order $2n$ with bipartition (A, B). If $d(x) + d(y) \geq n+1$ for any $x \in A$ and any $y \in B$ with $xy \notin E(G)$, then G is Hamiltonian.*

Lemma 2.4 (see [3]). *Let m, M, and γ_k $(k = 1, 2, \cdots, n)$ be real numbers satisfying $0 < m \leq \gamma_k \leq M$. Then*

$$\sum_{k=1}^{n} \gamma_k \sum_{k=1}^{n} \frac{1}{\gamma_k} \leq \frac{(m+M)^2}{4mM} n^2. \tag{1}$$

If $M > m$, then the equality sign in (1) holds if and only if n is an even; while, at the same time, for $n/2$ values of k one has $\gamma_k = m$ and for the remaining $n/2$ values of k one has $\gamma_k = M$. If $M = m$, then the equality in (1) always holds.

Notice that Lemma 2.4 is Corollary 4 on Page 67 in [3]; also, see [6].

3. Proofs

Proof of Theorem 1.1. Let G be a k-connected $(k \geq 2)$ graph with $n \geq 3$ vertices and e edges satisfying the conditions in Theorem 1.1. Suppose that G is not Hamiltonian. Then Lemma 2.1 implies that $\beta \geq k+1$. Also, we have that

$$n \geq 2\delta + 1 \geq 2k + 1,$$

otherwise $\delta \geq k \geq n/2$ and G is Hamiltonian. Let $I_1 := \{ u_1, u_2, ..., u_\beta \}$ be a maximum independent set in G. Then $I := \{ u_1, u_2, ..., u_{k+1} \}$ is an independent set in G. Thus

$$\sum_{u \in I} d(u) = |E(I, V - I)| \leq \sum_{v \in V - I} d(v).$$

Since

$$\sum_{u \in I} d(u) + \sum_{v \in V - I} d(v) = 2e,$$

we have that

$$\sum_{u \in I} d(u) \leq e \leq \sum_{v \in V - I} d(v).$$

Let $V - I = \{ v_1, v_2, ..., v_{n-(k+1)} \}$. By Cauchy-Schwarz inequality, we have

$$\sum_{r=1}^{n-(k+1)} 1^2 \sum_{r=1}^{n-(k+1)} d^2(v_r) \geq \left(\sum_{r=1}^{n-(k+1)} d(v_r) \right)^2 \geq e^2.$$

Thus,

$$\sum_{r=1}^{n-(k+1)} d^2(v_r) \geq \frac{e^2}{n-(k+1)}.$$

Consequently, we obtain

$$M := (k+1)\delta^2 + \frac{e^2}{n-(k+1)} \leq \sum_{u \in I} d^2(u) + \sum_{v \in V - I} d^2(v) = \sum_{v \in V} d^2(v).$$

Notice that $0 < 2\delta \leq d(u) + d(v) \leq 2\Delta$ for each edge uv in G. By Lemma 2.4, we have

$$\frac{(2\delta + 2\Delta)^2}{4\,(2\delta)\,(2\Delta)} e^2 = M \frac{(\delta + \Delta)^2}{4M\delta\Delta} e^2 \leq \sum_{v \in V} d^2(v) \, \frac{H(G)}{2} = \sum_{uv \in E} (d(u) + d(v)) \sum_{uv \in E} \frac{1}{d(u) + d(v)} \leq \frac{(2\delta + 2\Delta)^2}{4\,(2\delta)\,(2\Delta)} e^2.$$

Hence, we have

$$M = \sum_{v\in V} d^2(v) = \sum_{u\in I} d^2(u) + \sum_{v\in V-I} d^2(v) = (k+1)\delta^2 + \frac{e^2}{n-(k+1)}$$

and

$$\sum_{uv\in E} (d(u)+d(v)) \sum_{uv\in E} \frac{1}{d(u)+d(v)} = \frac{(2\delta+2\Delta)^2}{4\,(2\delta)\,(2\Delta)} e^2.$$

Therefore,

$$\sum_{u\in I} d^2(u) = (k+1)\delta^2$$

and

$$\sum_{r=1}^{n-(k+1)} 1^2 \sum_{r=1}^{n-(k+1)} d^2(v_r) = \left(\sum_{r=1}^{n-(k+1)} d(v_r)\right)^2 = e^2.$$

So, $d(u_1) = d(u_2) = \cdots = d(u_{k+1}) = \delta$, $\delta_1 := d(v_1) = d(v_2) = \cdots = d(v_{n-(k+1)}) \geq \delta$, and $\sum_{v\in V-I} d(v) = e$ which implies that $\sum_{u\in I} d(u) = e$ and G is a bipartite graph with partition sets of I and $V - I$.

If $\delta = \Delta$, then $(k+1)\delta = (n-(k+1))\delta_1 = (n-(k+1))\delta$. Thus, $n = 2k+2$ and Lemma 2.3 implies that G is Hamiltonian, which is a contradiction.

Now, assume that $\delta < \Delta$. Since

$$\sum_{uv\in E} (d(u)+d(v)) \sum_{uv\in E} \frac{1}{d(u)+d(v)} = \frac{(2\delta+2\Delta)^2}{4\,(2\delta)\,(2\Delta)} e^2,$$

from Lemma 2.4, it follows that e must be even and there exists an edge xy such that $d(x) + d(y) = 2\delta$, where $x \in I$ and $y \in V-I$, and an edge zw such that $d(z)+d(w) = 2\Delta$, where $w \in I$ and $z \in V-I$. Hence, $2\delta = d(x)+d(y) = \delta+\delta_1 = \delta+\Delta > 2\delta$, which is a contradiction. This completes the proof of Theorem 1.1. □

The proof of Theorem 1.2 is similar to the proof of Theorem 1.1. For the sake of completeness, we still present a full proof of Theorem 1.2.

Proof of Theorem 1.2. Let G be a k-connected ($k \geq 1$) graph with $n \geq 9$ vertices and e edges satisfying the conditions in Theorem 1.2. Suppose that G is not traceable. Then Lemma 2.2 implies that $\beta \geq k+2$. Also, we have that $n \geq 2\delta+2 \geq 2k+2$ otherwise $\delta \geq k \geq (n-1)/2$ and G is traceable. Let $I_1 := \{ u_1, u_2, ..., u_\beta \}$ be a maximum independent set in G. Then, $I := \{ u_1, u_2, ..., u_{k+2} \}$ is an independent set in G. Thus,

$$\sum_{u\in I} d(u) = |E(I, V-I)| \leq \sum_{v\in V-I} d(v).$$

Since

$$\sum_{u\in I} d(u) + \sum_{v\in V-I} d(v) = 2e,$$

we have

$$\sum_{u\in I} d(u) \leq e \leq \sum_{v\in V-I} d(v).$$

Let $V - I = \{ v_1, v_2, ..., v_{n-(k+2)} \}$. By Cauchy-Schwarz inequality, we have

$$\sum_{r=1}^{n-(k+2)} 1^2 \sum_{r=1}^{n-(k+2)} d^2(v_r) \geq \left(\sum_{r=1}^{n-(k+2)} d(v_r)\right)^2 \geq e^2.$$

Thus,

$$\sum_{r=1}^{n-(k+2)} d^2(v_r) \geq \frac{e^2}{n-(k+2)}.$$

Therefore,

$$N := (k+2)\delta^2 + \frac{e^2}{n-(k+2)} \leq \sum_{u\in I} d^2(u) + \sum_{v\in V-I} d^2(v) = \sum_{v\in V} d^2(v).$$

Notice that $0 < 2\delta \leq d(u) + d(v) \leq 2\Delta$ for every edge uv in G. By Lemma 2.4, we have

$$\frac{(2\delta + 2\Delta)^2}{4\,(2\delta)\,(2\Delta)}e^2 = N\frac{(\delta + \Delta)^2}{4N\delta\Delta}e^2 \leq \sum_{v\in V} d^2(v)\,\frac{H(G)}{2} = \sum_{uv\in E}(d(u)+d(v))\sum_{uv\in E}\frac{1}{d(u)+d(v)} \leq \frac{(2\delta + 2\Delta)^2}{4\,(2\delta)\,(2\Delta)}e^2.$$

Thus,

$$N = \sum_{v\in V} d^2(v) = \sum_{v\in I} d^2(v) + \sum_{v\in V-I} d^2(v) = (k+2)\delta^2 + \frac{e^2}{n-(k+2)}$$

and

$$\sum_{uv\in E}(d(u)+d(v))\sum_{uv\in E}\frac{1}{d(u)+d(v)} = \frac{(2\delta + 2\Delta)^2}{4\,(2\delta)\,(2\Delta)}e^2.$$

Therefore,

$$\sum_{v\in I} d^2(v) = (k+2)\delta^2$$

and

$$\sum_{r=1}^{n-(k+2)} 1^2 \sum_{r=1}^{n-(k+2)} d^2(v_r) = \left(\sum_{r=1}^{n-(k+2)} d(v_r)\right)^2 = e^2.$$

So $d(u_1) = d(u_2) = \cdots = d(u_{k+2}) = \delta$, $\delta_1 := d(v_1) = d(v_2) = \cdots = d(v_{n-(k+2)}) \geq \delta$, and $\sum_{v\in V-I} d(v) = e$, which implies that $\sum_{u\in I} d(u) = e$ and G is a bipartite graph with partition sets of I and $V - I$.

If $\delta = \Delta$, then $(k+2)\delta = (n-(k+2))\delta_1 = (n-(k+2))\delta$. Thus, $n = 2k+4$. Since $n = 2k+4 \geq 9$, we have that $k \geq 3$. Thus, Lemma 2.3 implies that G is Hamiltonian and therefore G is traceable, which is a contradiction.

Next, we assume that $\delta < \Delta$. Since

$$\sum_{uv\in E}(d(u)+d(v))\sum_{uv\in E}\frac{1}{d(u)+d(v)} = \frac{(2\delta + 2\Delta)^2}{4\,(2\delta)\,(2\Delta)}e^2,$$

from Lemma 2.4 it follows that e must be even and there must exist an edge xy such that $d(x)+d(y) = 2\delta$, where $x \in I$ and $y \in V-I$, and an edge zw such that $d(z)+d(w) = 2\Delta$, where $w \in I$ and $z \in V-I$. Hence, $2\delta = d(x)+d(y) = \delta+\delta_1 = \delta+\Delta > 2\delta$, which is again a contradiction. This completes the proof of Theorem 1.2. □

From the proofs of Theorem 1.1 and Theorem 1.2, the following corollary is obtained:

Corollary 3.1. *Let G be a graph with n vertices and $e \geq 1$ edges. Then*

$$H(G) \leq \frac{(\delta+\Delta)^2}{2Q\delta\Delta}e^2, \tag{2}$$

where

$$Q = \beta\delta^2 + \frac{e^2}{n-\beta}.$$

The equality in (2) holds if and only if G is a regular balanced bipartite graph.

Proof of Corollary 3.1. Let $I := \{u_1, u_2, ..., u_\beta\}$ be a maximum independent set in G. Since $e \geq 1$, we have $|I| < n$. Thus, $|V - I| > 0$. From the proof of Theorem 1.1, we have

$$Q := \beta\delta^2 + \frac{e^2}{n-\beta} \leq \sum_{u\in I} d^2(u) + \sum_{v\in V-I} d^2(v) = \sum_{v\in V} d^2(v)$$

and

$$H(G) = \sum_{uv\in E}\frac{2}{d(u)+d(v)} \leq \frac{2(2\delta+2\Delta)^2}{4Q(2\delta)(2\Delta)}e^2 = \frac{(\delta+\Delta)^2}{2Q\delta\Delta}e^2.$$

If

$$H(G) = \frac{(\delta+\Delta)^2}{2Q\delta\Delta}e^2,$$

then from the proof of Theorem 1.1, it follows that G is bipartite with two partition sets of I and $V - I$ such that $d(u) = \delta$ for each vertex $u \in I$, $d(v) = \delta_1$ for each vertex $v \in V - I$, and $e = \beta\delta = (n-\beta)\delta_1$. If $\delta = \Delta$, then $\beta\delta = (n-\beta)\delta_1 = (n-\beta)\delta$. Thus, $n = 2\beta$ and G is a regular balanced bipartite graph.

If $\delta < \Delta$, then Lemma 2.4 implies that e is even and there exists an edge xy, where $x \in I$ and $y \in V - I$, such that $d(x)+d(y) = 2\delta$, and an edge zw, where $z \in I$ and $w \in V-I$, such that $d(z)+d(w) = 2\Delta$. Since $d(x)+d(y) = \delta+\delta_1 = \delta+\Delta > 2\delta$, which is a contradiction.

If G is a regular balanced bipartite graph, simple computations yield

$$H(G) = \frac{(\delta + \Delta)^2}{2Q\delta\Delta}e^2 = \frac{n}{2}.$$

This completes the proof of Corollary 3.1. □

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