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Research Article

Domination number of Cartesian product through space projections

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Abstract

In 1968, Vizing conjectured that for every pair of graphs X and Y, the inequality $\gamma(X\Box Y) \geq \gamma(X)\gamma(Y)$ holds, where γ stands for the domination number and $X\Box Y$ is the Cartesian product of X and Y. In a breakthrough result, Clark and Suen [*Electron. J. Combin.* **7** (2000) #N4] proved that $\gamma(X\Box Y) \geq \frac{1}{2}\gamma(X)\gamma(Y)$. In this paper, a lower bound for $\gamma(X\Box Y\Box Z)$ is obtained using projections in the space. It is shown how the obtained bound implies the mentioned result of Clark and Suen.

Keywords: Cartesian product; domination number; Vizing's conjecture; Clark-Suen bound.

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1. Introduction

Let G be a simple finite graph and let $V(G)$ be its set of vertices. We say that a vertex $u \in V(G)$ dominates a vertex v if $u = v$ or v is adjacent to u. A dominating set of G is a subset S of $V(G)$ whose vertices dominate all the vertices of G. The domination number of G, denoted as $\gamma(G)$, is the size of a smallest dominating set of G.

The Cartesian product $X\Box Y$ of two graphs X and Y is the graph whose vertex set is $V(X) \times V(Y)$ and whose edge set is defined as follows: Two vertices (x_1, y_1) and (x_2, y_2) are adjacent in $X\Box Y$ if either $x_1 = x_2$ and y_1 and y_2 are adjacent in Y, or $y_1 = y_2$ and x_1 and x_2 are adjacent in X. By definition, the Cartesian product of graphs is commutative, in the sense that $X\Box Y$ is isomorphic to $Y\Box X$. For $y \in V(Y)$, the subgraph of $X\Box Y$ induced by $\{(x, y) \mid x \in V(X)\}$, is called an X-fiber and denoted as X^y.

In 1968, Vizing conjectured in [6] that for every pair of graphs X and Y, $\gamma(X\Box Y) \geq \gamma(X)\gamma(Y)$. Since then, many authors have found weaker lower bounds for $\gamma(X\Box Y)$. For example, Clark and Suen showed in [2], that $\gamma(X\Box Y) \geq \frac{1}{2}\gamma(X)\gamma(Y)$. This lower bound was improved by Suen and Tarr in [4], and recently by Zerbib in [7].

The result of Clark and Suen implies that for a triple of graphs, X, Y and Z, we have $\gamma(X\Box Y\Box Z) \geq \frac{1}{2}\gamma(X\Box Y)\gamma(Z) \geq \frac{1}{4}\gamma(X)\gamma(Y)\gamma(Z)$. The main result of this paper, Theorem 2.1, provides a lower bound for $\gamma(X\Box Y\Box Z)$ which can be used to re-obtain the mentioned result of Clark and Suen. To prove it, we use a modification of the new framework, to approach Vizing's conjecture, developed in [1].

2. Results

Let X and Y be graphs with $\gamma(X) = k$ and $\gamma(Y) = r$. Consider $\{u_1, \cdots, u_k\}$ to be a minimum dominating set of X and $\{v_1, \cdots, v_r\}$ to be a minimum dominating set of Y. Let $\pi = \{\pi_{i,j}; 1 \leq i \leq k, 1 \leq j \leq r\}$ be a partition of $V(X\Box Y)$ chosen so that $(u_i, v_j) \in \pi_{i,j}$ and $\pi_{i,j} \subseteq N[u_i] \times N[v_j]$ for any $1 \leq i \leq k$ and $1 \leq j \leq r$, where $N[x]$ is the set containing x and all the vertices adjacent to it. Let Z be a graph and define $Z_{i,j} := \pi_{i,j} \times V(Z)$. For a vertex $z \in V(Z)$, the set of vertices $\pi_{i,j} \times \{z\}$ is called a cell, and is denoted $\pi^z_{i,j}$. We may say that the cell $\pi^z_{i,j}$ belongs to $Z_{i,j}$, and from the other perspective it also belongs to the fiber $(X\Box Y)^z$.

For a natural number k, the set $\{1, \cdots, k\}$ we be denote by $[k]$. Let D be a minimum dominating set of $X\Box Y\Box Z$. For $(i, j) \in [k] \times [r]$, let $D_{i,j} = D \cap Z_{i,j}$. Similarly, we denote $D^z = D \cap (X\Box Y)^z$ for $z \in V(Z)$. We color the cell $\pi^z_{i,j}$ blue if $\pi^z_{i,j} \cap D \neq \emptyset$ and $\pi^z_{i,j}$ is dominated by D^z. The cell $\pi^z_{i,j}$ is colored green if $\pi^z_{i,j} \cap D \neq \emptyset$ and $\pi^z_{i,j}$ is not dominated by D^z. Finally, we color the cell red if it is dominated by D^z and no vertex of $\pi^z_{i,j}$ is dominated by $D_{i,j}$. All the remaining cells in $(X\Box Y)^z$ are colored white. Note that exactly the cells with color blue and green contain vertices of D. In the set $V(X\Box Y\Box Z)$ we only color the vertices of D as follows. The vertices in $D \cap \pi^z_{i,j}$ are colored blue (resp. green) if the cell $\pi^z_{i,j}$ is blue (resp. green). This coloring of the vertices of D is a partition of D into subsets of blue vertices and green vertices.

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Example 2.1. *Consider the three graphs X, Y and Z with vertex sets $V(X) = \{x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8\}$, $V(Y) = \{y_1, y_2, y_3, y_4, y_5\}$, $V(Z) = \{z_1, z_2, z_3\}$ respectively and edge sets defined as follows*

$$E(X) = \{x_1x_2, x_1x_3, x_2x_3, x_2x_6, x_3x_4, x_4x_5, x_5x_6, x_5x_8, x_6x_7, x_7x_8\},$$

$$E(Y) = \{y_1y_2, y_1y_4, y_2y_3, y_2y_5, y_3y_4, y_4y_5\},$$

and

$$E(Z) = \{z_1z_2, z_2z_3\}.$$

It would be easy to check that the domination number of X is 3, of Y is 2 and that of Z is 1. Consider the dominating sets $\{x_2, x_5, x_7\}$ and $\{y_2, y_4\}$ of X and Y respectively. Let $\pi = \{\pi_{1,1}, \pi_{1,2}, \pi_{2,1}, \pi_{2,2}, \pi_{3,1}, \pi_{3,2}\}$ be the partition of $V(X\Box Y)$ where:

$$\pi_{1,1} = \{(x_1, y_1), (x_2, y_1), (x_3, y_1), (x_1, y_2), (x_2, y_2), (x_3, y_2)\},$$

$$\pi_{1,2} = \{(x_1, y_3), (x_2, y_3), (x_3, y_3), (x_1, y_4), (x_2, y_4), (x_3, y_4), (x_1, y_5), (x_2, y_5), (x_3, y_5)\},$$

$$\pi_{2,1} = \{(x_4, y_1), (x_5, y_1), (x_4, y_2), (x_5, y_2)\},$$

$$\pi_{2,2} = \{(x_4, y_3), (x_5, y_3), (x_4, y_4), (x_5, y_4), (x_6, y_4), (x_4, y_5), (x_5, y_5)\},$$

$$\pi_{3,1} = \{(x_6, y_1), (x_7, y_1), (x_8, y_1), (x_6, y_2), (x_7, y_2), (x_8, y_2), (x_6, y_3), (x_7, y_3), (x_8, y_3),$$
$$(x_6, y_5), (x_7, y_5), (x_8, y_5)\},$$

$$\pi_{3,2} = \{(x_7, y_4), (x_8, y_4)\}.$$

We used SageMath [3] to verify that the domination number of $X\Box Y\Box Z$ is 19 and that a minimal dominating set is given by

$$D = \{(x_1, y_4, z_1), (x_2, y_2, z_1), (x_4, y_2, z_1), (x_5, y_4, z_1), (x_6, y_4, z_1), (x_8, y_2, z_1), (x_3, y_1, z_2), (x_3, y_3, z_2),$$
$$(x_3, y_5, z_2), (x_5, y_2, z_2), (x_7, y_1, z_2), (x_7, y_3, z_2), (x_7, y_5, z_2), (x_1, y_2, z_3), (x_2, y_4, z_3),$$
$$(x_4, y_4, z_3), (x_5, y_2, z_3), (x_6, y_2, z_3), (x_8, y_4, z_3)\}.$$

In Figure 2.1, we show the coloring of the cells in blue, green, red and white (cells with dashed edges) as well as the coloring of the vertices of D in blue and green. The cells in the bottom correspond to the layer $(X\Box Y)^{z_1}$, those in the middle correspond to the layer $(X\Box Y)^{z_2}$, while on the top figures the cells of the layer $(X\Box Y)^{z_3}$. The reader is invited to check that the colors given to cells agrees with our construction. For example, the cell $\pi_{2,1}^{z_1}$ is colored blue since it contains a vertex of D and all its vertices are dominated by $D \cap (X\Box Y)^{z_1}$ while $\pi_{1,1}^{z_1}$ is colored green since it contains a vertex of D but the vertex $(x_3, y_1, z_1) \in \pi_{1,1}^{z_1}$ and is not dominated by any vertex in $D \cap (X\Box Y)^{z_1}$. Also, $\pi_{3,2}^{z_1}$ is colored red because it contains no vertex of D and all of its vertices are dominated by $D \cap (X\Box Y)^{z_1}$, but $\pi_{2,2}^{z_2}$ is colored white because it contains no vertex of D and some of its vertices, (x_4, y_4, z_2) for example, is not dominated by $D \cap (X\Box Y)^{z_2}$.

Now let b'_z be the number of blue cells in the fiber $(X\Box Y)^z$ and b' be the total number of blue cells in $X\Box Y\Box Z$. We define analogously g'_z and g' associated with the green cells, r'_z and r' associated with red cells and w'_z and w' associated with the white cells. Also, let b_z denote the number of blue vertices in $(X\Box Y)^z$ and let b be the total number of blue vertices in $X\Box Y\Box Z$. In an analogous way define g_z and g, associated with the green vertices. It would be clear that $b_z \geq b'_z$, $b \geq b'$, $g_z \geq g'_z$, $g \geq g'$ and $|D| = b + g$.

Lemma 2.1. $b' + g' + r' \geq \gamma(X)\gamma(Y)\gamma(Z)$.

Proof. For any $(i, j) \in [k] \times [r]$, in the projection p_Z of the cells of $Z_{i,j}$ to Z, let the vertex $z = p_Z(\pi_{i,j}^z)$ receive the color of the cell $\pi_{i,j}^z$. Let $B_{i,j}, G_{i,j}$, $R_{i,j}$ and $W_{i,j}$ be the resulting set of vertices in Z colored blue, green, red and white respectively, for each $(i, j) \in [k] \times [r]$. Since in each white cell $\pi_{i,j}^z$, there should be a vertex covered vertically by a blue or a green vertex, then it would be clear that the disjoint union $B_{i,j} \sqcup G_{i,j} \sqcup R_{i,j}$ dominates Z. That is $|B_{i,j}| + |G_{i,j}| + |R_{i,j}| \geq \gamma(Z)$ for any $(i, j) \in [k] \times [r]$. Thus,

$$b' + g' + r' = \sum_{i=1}^{k}\sum_{j=1}^{r}(|B_{i,j}| + |G_{i,j}| + |R_{i,j}|) \geq \sum_{i=1}^{k}\sum_{j=1}^{r}\gamma(Z) = \gamma(X)\gamma(Y)\gamma(Z).$$

□

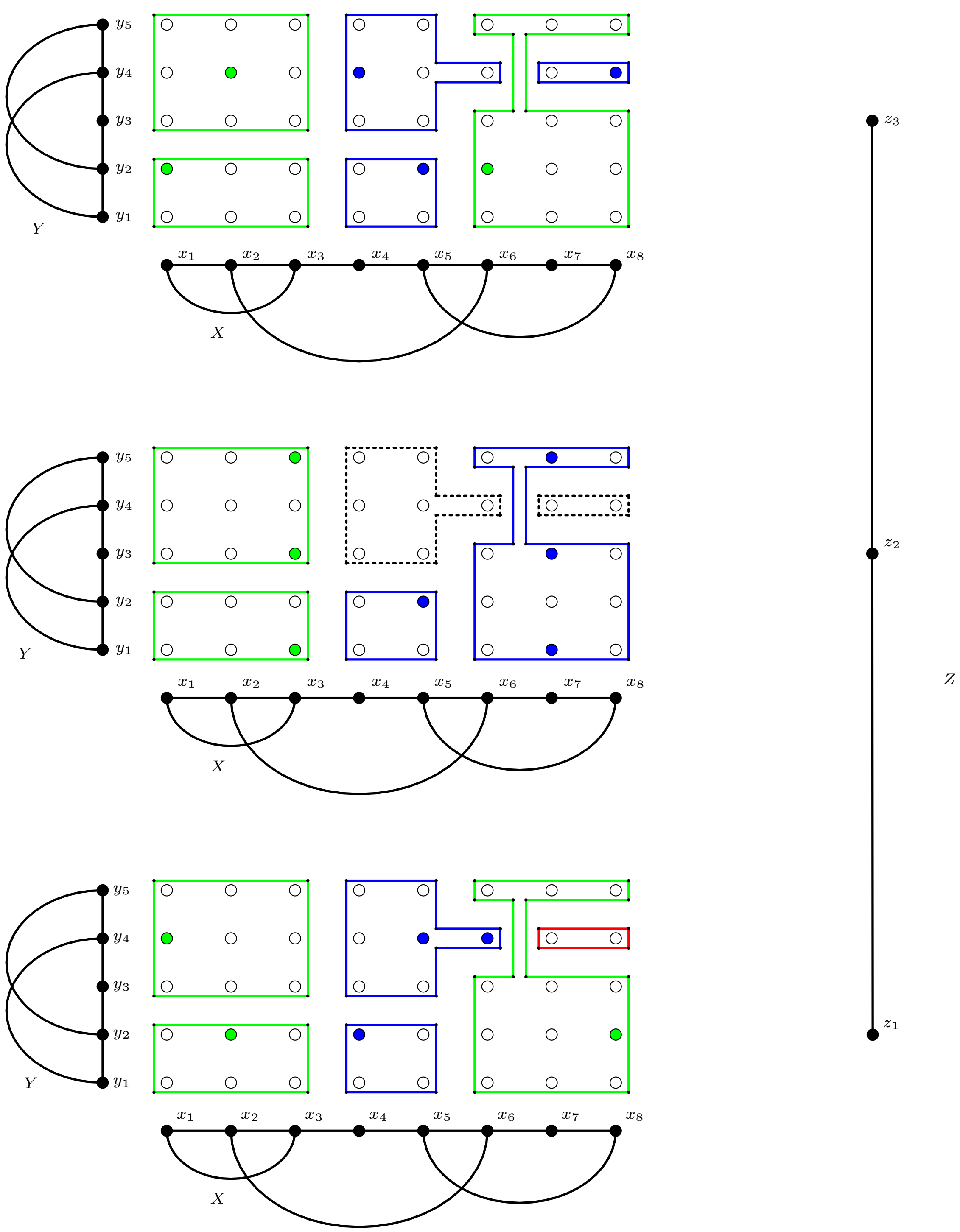

Figure 2.1: Cell coloring for the cartesian product $X \square Y \square Z$.

Example 2.2. *Following with Example 2.1, one can check that for any* $(i,j) \in [3] \times [2]$, $B_{i,j} \sqcup G_{i,j} \sqcup R_{i,j}$ *dominates* Z. *For example,* $B_{2,2} \sqcup G_{2,2} \sqcup R_{2,2}$ *and* $B_{3,2} \sqcup G_{3,2} \sqcup R_{3,2}$ *both dominate* Z. *In addition,* $b'+g'+r' = 7+8+1 \geq \gamma(X)\gamma(Y)\gamma(Z) = 3{\cdot}2{\cdot}1 = 6$.

For each cell $\pi_{i,j}$, we define $l_Y(\pi_{i,j})$ to be $|p_Y(\pi_{i,j})|$, where p_Y is the projection map from $V(X \square Y)$ to $V(Y)$. Similarly, define $l_X(\pi_{i,j})$ to be $|p_X(\pi_{i,j})|$. Now define $l(\pi_{i,j})$ to be the minimum of $l_X(\pi_{i,j})$ and $l_Y(\pi_{i,j})$,

$$l(\pi_{i,j}) := \min\{l_X(\pi_{i,j}), l_Y(\pi_{i,j})\}$$

and let $L_{X,Y}$ be the maximum of all $l(\pi_{i,j})$,

$$L_{X,Y} := \max\{l(\pi_{i,j}); 1 \leq i \leq k, 1 \leq j \leq r\}.$$

Example 2.3. *In Example 2.1, we have* $l(\pi_{1,1}) = 2$, $l(\pi_{1,2}) = 3$, $l(\pi_{2,1}) = 2$, $l(\pi_{2,2}) = 3$, $l(\pi_{3,1}) = 3$ *and* $l(\pi_{3,2}) = 1$. *Therefore,* $L_{X,Y} = 3$.

By definition, $l(\pi_{i,j}) \leq \min\{\deg_X(u_i) + 1, \deg_Y(v_j) + 1\}$, where $\deg_X(u_i)$ is the degree of the vertex u_i in X, and

$$L_{X,Y} \leq \min\{\Delta(X) + 1, \Delta(Y) + 1\},$$

where $\Delta(X)$ is the maximum vertex degree in X.

Lemma 2.2. $L_{X,Y} r' \leq b + g - L_{X,Y} b' + |V(Z)| \left(L_{X,Y}\gamma(X)\gamma(Y) - \gamma(X \square Y)\right)$.

Proof. For any $z \in V(Z)$, in the fiber $(X \square Y)^z$, the blue and green vertices dominate all the vertices in the blue and red cells. To dominate all the vertices in the green and white cells, it would be enough, for each green or white cell $\pi^z_{i,j}$, to consider the vertices $\{(u_i, t, z) : t \in p_Y(\pi_{i,j})\}$ if $l(\pi_{i,j})) = l_Y(\pi_{i,j})$ or the vertices $\{(t, v_j, z) : t \in p_X(\pi_{i,j})\}$ if $l(\pi_{i,j})) = l_X(\pi_{i,j})$. Therefore, for any $z \in V(Z)$, we have

$$\gamma(X \square Y) = \gamma((X \square Y)^z) \leq b_z + g_z + \sum_{\pi \in G_z \sqcup W_z} l(\pi),$$

where G_z and W_z denote respectively the set of green and white cells in $(X \square Y)^z$. Using the fact that $l(\pi) \leq L_{X,Y}$ for any $\pi \in G_z \sqcup W_z$, we get:

$$\gamma(X \square Y) \leq b_z + g_z + L_{X,Y}(g'_z + w'_z).$$

for any $z \in V(Z)$. But for any $z \in V(Z)$, the total number of cells in $(X \square Y)^z$ is $b'_z + g'_z + w'_z + r'_z = \gamma(X)\gamma(Y)$. Therefore, for any $z \in V(Z)$, we have

$$\gamma(X \square Y) \leq b_z + g_z + L_{X,Y}(\gamma(X)\gamma(Y) - b'_z - r'_z).$$

By summing over all the vertices $z \in V(Z)$, we obtain:

$$|V(Z)|\gamma(X \square Y) \leq b + g + L_{X,Y}(|V(Z)|\gamma(X)\gamma(Y) - b' - r').$$

The result follows. $\square$

Remark 2.1. *From the proof of Lemma 2.2, one can improve the definition of* $L_{X,Y}$ *and take it to be*

$$\max\{l(\pi); \pi \in G_z \sqcup W_z \text{ for any } z \in V(Z)\}.$$

However, this will not affect the results which will follow.

Theorem 2.1. *For every triple of graphs* X, Y *and* Z, *it holds that* $\gamma(X \square Y \square Z) \geq \alpha^Z_{X,Y}$, *where*

$$\alpha^Z_{X,Y} := \frac{L_{X,Y}}{L_{X,Y}+1}\gamma(X)\gamma(Y)\gamma(Z) - \frac{|V(Z)|}{L_{X,Y}+1}\left(L_{X,Y}\gamma(X)\gamma(Y) - \gamma(X \square Y)\right).$$

Proof. By Lemmas 2.1 and 2.2, we have

$$\begin{aligned} L_{X,Y}\gamma(X)\gamma(Y)\gamma(Z) &\leq L_{X,Y}b' + L_{X,Y}g' + L_{X,Y}r' \\ &\leq L_{X,Y}g' + b + g + |V(Z)|\left(L_{X,Y}\gamma(X)\gamma(Y) - \gamma(X \square Y)\right). \end{aligned}$$

Using the facts that $b + g = |D| = \gamma(X \square Y \square Z)$ and $g' \leq g \leq |D|$, we get

$$L_{X,Y}\gamma(X)\gamma(Y)\gamma(Z) \leq (L_{X,Y}+1)\gamma(X \square Y \square Z) + |V(Z)|\left(L_{X,Y}\gamma(X)\gamma(Y) - \gamma(X \square Y)\right).$$

The result follows. $\square$

Using the fact that the Cartesian product of graphs is commutative and associative, we have the following result.

Corollary 2.1. *For every triple of graphs X, Y and Z, the following inequality holds:*

$$\gamma(X\square Y\square Z) \geq \max\{\alpha^Z_{X,Y}, \alpha^Y_{X,Z}, \alpha^X_{Y,Z}\}.$$

Remark 2.2. *In the lower bound obtained in Theorem 2.1, $L_{X,Y}$ depends on the choice of the partition of $V(X\square Y)$. However, as it will next appear, for some graphs X with a few number of vertices, the value of $L_{X,Y}$ will be limited to 1 or 2.*

Example 2.4. *In this example, we fix $X = P_2$, $Y = P_3$ and $Z = P_4$, where P_2, P_3 and P_4 are the path graphs on two, three and four vertices respectively. As stated in Remark 2.2, in general $L_{X,Y}$ depends on the choice of the partition of $V(X\square Y)$. But obviously, taking the (trivial) partition of $V(P_2\square P_3)$ which consists of putting all the vertices of $V(P_2\square P_3)$ in one cell implies that $L_{P_2,P_3} = 2$. In addition, it can be easily seen that one may take partitions of $V(P_2\square P_4)$ and $V(P_3\square P_4)$ so that $L_{P_2,P_4} = 2$ and $L_{P_3,P_4} = 2$. Since, $\gamma(P_2\square P_3) = 2$, $\gamma(P_2\square P_4) = 3$ and $\gamma(P_3\square P_4) = 4$, by Theorem 2.1, we have:*

$$\begin{aligned}
\alpha^{P_4}_{P_2,P_3} &= \frac{2}{2+1}1\cdot 1\cdot 2 - \frac{4}{2+1}(2\cdot 1\cdot 1 - 2) = \frac{4}{3},\\
\alpha^{P_3}_{P_2,P_4} &= \frac{2}{2+1}1\cdot 2\cdot 1 - \frac{3}{2+1}(2\cdot 1\cdot 2 - 3) = \frac{1}{3},\\
\alpha^{P_2}_{P_3,P_4} &= \frac{2}{2+1}1\cdot 2\cdot 1 - \frac{2}{2+1}(2\cdot 1\cdot 2 - 4) = \frac{4}{3}.
\end{aligned}$$

This implies by the Corollary 2.1 that $\gamma(P_2\square P_3\square P_4) \geq 2$.

Corollary 2.2 (Clark and Suen, [2])**.** *For any two graphs Y and Z, the following inequality holds:*

$$\gamma(Y\square Z) \geq \frac{1}{2}\gamma(Y)\gamma(Z).$$

Proof. Applying Theorem 2.1 for the trivial graph X, it would be clear that $\gamma(X) = 1$ and $L_{X,Y} = 1$. As such,

$$\gamma(X\square Y\square Z) = \gamma(Y\square Z) \geq \frac{1}{2}\gamma(Y)\gamma(Z) - \frac{|V(Z)|}{2}(\gamma(Y) - \gamma(Y)) = \frac{1}{2}\gamma(Y)\gamma(Z).$$

□

In [5], after a major change to the projection method developed in this paper, it is shown that for any pair of graphs X and Y, the inequality $\gamma(X\square Y\square P_2) \geq \frac{2}{3}\gamma(X)\gamma(Y)$ holds; and in general, it holds that $\gamma(X\square Y\square P_n) \geq c_n\gamma(X)\gamma(Y)\gamma(P_n)$, where c_n is almost $\frac{3}{4}$ when n is big enough.

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