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Research Article

Multicolor Ramsey theory for a fan versus complete graphs

Mark Budden*, Hayden Privette

Department of Mathematics and Computer Science, Western Carolina University, Cullowhee, North Carolina 28723, USA

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Abstract

Let mK_2 be the disjoint union of m copies of K_2 and define the fan $F_m := K_1 + mK_2$. The multicolor Ramsey number, $r(G_1, G_2, \ldots, G_t)$ is the least natural number p such that any t-coloring of K_p contains a monochromatic G_i in some color i. The star-critical Ramsey number is denoted by $r_*(G_1, G_2. \ldots, G_t)$. We consider the case of fans versus complete graphs. We show that $r(F_m, K_3, K_3) = 10m + 1$ and $r_*(F_m, K_3, K_3) \geq 8m + 2$, for all $m \geq 6$. We also examine the corresponding Gallai-Ramsey numbers, proving that $gr(F_2, K_3, K_3) = 21$ and $gr_*(F_2, K_3, K_3) = 18$.

Keywords: Gallai-Ramsey number; critical colorings; star-critical Ramsey number.

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1. Introduction

Let K_n denote a complete graph of order n and let mK_n be the disjoint union of m copies of K_n. The *join* of two graphs G_1 and G_2, denoted $G_1 + G_2$, is the graph formed by taking the disjoint union of G_1 and G_2 and adding in all edges that include a single vertex from G_1 and a single vertex from G_2. The *fan* F_m is defined by $F_m := K_1 + mK_2$ and the vertex that corresponds with the K_1-subgraph is called the *center vertex*. We note that F_m has order $2m + 1$ and contains m *blades* (corresponding with the m disjoint K_2-subgraphs). Since $F_1 \cong K_3$, any time we consider a fan F_m, it is assumed that $m \geq 2$.

A *t-coloring* of a graph G is a map $c : E(G) \longrightarrow \{1, 2, \ldots, t\}$, where the numbers in $\{1, 2, \ldots, t\}$ are identified with colors. We do not assume that such a map is surjective, so a t-coloring may use fewer than t colors. For graphs $G_1, G_2, \ldots, G_t$, the *Ramsey number* $r(G_1, G_2, \ldots, G_t)$ is the least $p \in \mathbb{N}$ such that every t-coloring of K_p contains a monochromatic subgraph isomorphic to G_i in color i, for some $i \in \{1, 2, \ldots, t\}$. When $r(G_1, G_2, \ldots, G_t) = p$, a t-coloring of K_{p-1} that avoids a monochromatic copy of G_i in color i, for all $i \in \{1, 2, \ldots, t\}$, is called a *critical coloring for* $r(G_1, G_2, \ldots, G_t)$.

Ramsey numbers for fans (containing at least 2 blades) versus complete graphs have been considered in the following papers:

$$r(F_m, K_3) = 4m + 1, \quad \text{for all } m \geq 2 \ [15],$$
$$r(F_m, K_4) = 6m + 1, \quad \text{for all } m \geq 4 \ [18],$$
$$r(F_m, K_5) = 8m + 1, \quad \text{for all } m \geq 5 \ [20],$$
$$r(F_m, K_6) = 10m + 1, \quad \text{for all } m \geq 6 \ [13].$$

In [18], Surahmat, Baskoro, and Broersma gave the following conjecture.

Conjecture 1.1 (see [18])**.** *For all* $m \geq n \geq 3$*,* $r(F_m, K_n) = 2m(n - 1) + 1$.

Besides the above-known cases of this conjecture, a result by Li and Rousseau [15] implies that this conjecture is true for sufficiently large fans. In this paper, we consider the analogous problem for more than two colors and also consider the evaluation of the corresponding star-critical Ramsey numbers.

In order to describe a star-critical Ramsey number, define the notation $K_n \sqcup K_{1,k}$ to be the graph formed by taking the disjoint union of K_n and a single vertex, then adding in exactly k edges ($1 \leq k \leq n$) between the single vertex and the complete graph. The *star-critical Ramsey number* $r_*(G_1, G_2, \ldots, G_t)$ is then defined to be the least k such that every t-coloring of $K_{r(G_1,G_2,\ldots,G_t)-1} \sqcup K_{1,k}$ contains a monochromatic G_i in color i, for some $i \in \{1, 2, \ldots, t\}$. Star-critical Ramsey

*Corresponding author (mrbudden@email.wcu.edu).

numbers were first defined by Jonelle Hook in her dissertation [12]. The known star-critical Ramsey numbers for fans (containing at least 2 blades) versus complete graphs are

$$r_*(F_m, K_3) = 2m + 2, \quad \text{for all } m \geq 2 \ [14],$$
$$r_*(F_m, K_4) = 4m + 2, \quad \text{for all } m \geq 4 \ [9].$$

In Section 2, we focus on proving lower bound results for multicolor Ramsey and star-critical Ramsey numbers, and we consider their application to fans versus complete graphs. In particular, we show that

$$r(F_m, K_3, K_3) = 10m + 1 \quad \text{and} \quad r_*(F_m, K_3, K_3) \geq 8m + 2,$$

for all $m \geq 6$. Unfortunately, these results do not immediately extend to the cases $2 \leq m \leq 5$. In order to make progress when $m = 2$, in Section 3, we turn our attention to Gallai t-colorings.

A *Gallai t-coloring* of a graph G is a t-coloring of G that avoids rainbow triangles (K_3-subgraphs in which the three edges receive distinct colors). The *Gallai-Ramsey number* $gr(G_1, G_2, \ldots, G_t)$ is the least $p \in \mathbb{N}$ such that every Gallai t-coloring of K_p contains a monochromatic copy of G_i in color i, for some $i \in \{1, 2, \ldots, t\}$. A Gallai t-coloring of $K_{gr(G_1,G_2,\ldots,G_t)-1}$ that avoids a monochromatic copy of G_i in color i, for all $i \in \{1, 2, \ldots, t\}$, is called a *critical coloring for* $gr(G_1, G_2, \ldots, G_t)$. Since every Gallai t-coloring is a t-coloring, it follows that

$$gr(G_1, G_2, \ldots, G_t) \leq r(G_1, G_2, \ldots, G_t).$$

The *star-critical Gallai-Ramsey number* $gr_*(G_1, G_2, \ldots, G_t)$ is the least k such that every Gallai t-coloring of

$$K_{gr(G_1,G_2,\ldots,G_t)-1} \sqcup K_{1,k}$$

contains a monochromatic copy of G_i in color i, for some $i \in \{1, 2, \ldots, t\}$. Besides the general inequality

$$gr(F_m, \underbrace{K_3, \ldots, K_3}_{t \text{ terms}}) \geq \begin{cases} 2m \cdot 5^{t/2} + 1 & \text{if } t \text{ is even} \\ 4m \cdot 5^{(t-1)/2} + 1 & \text{if } t \text{ is odd,} \end{cases}$$

for all $m \geq 2$, we prove that

$$gr(F_2, K_3, K_3) = 21 \quad \text{and} \quad gr_*(F_2, K_3, K_3) = 18.$$

In Section 4, we conclude by providing some additional conjectures that we hope will guide future research.

2. Lower bounds in Ramsey theory

For a graph $G = (V, E)$, denote by $c(G)$ the order of its largest connected component and let $\chi(G)$ be its chromatic number. So, $c(G) = |V(G)|$ whenever G is connected. In 1972, Chvátal and Harary [7] proved that

$$r(G_1, G_2) \geq (c(G_1) - 1)(\chi(G_2) - 1) + 1,$$

for any graphs G_1 and G_2 that lack isolated vertices. When equality holds and $G_2 = K_n$, we say that G_1 is *n-good*. This concept was introduced in 1983 by Burr and Erdős [5] in the case where G_2 is a complete graph, and has seen many generalizations (e.g., see [2], [3], and [4]). The following lemma offers an additional generalization.

Lemma 2.1. *Let $t \geq 3$ and assume that $n_i \geq 2$, for all $i \in \{2, \ldots, t\}$. Then for any graph G,*

$$r(G, K_{n_2}, \ldots, K_{n_t}) \geq (c(G) - 1)(r(K_{n_2}, \ldots, K_{n_t}) - 1) + 1.$$

Proof. Begin with a critical coloring for $r(K_{n_2}, \ldots, K_{n_t})$, which is a $(t-1)$-coloring of $K_{r(K_{n_2},\ldots,K_{n_t})-1}$, using colors $2, \ldots, t$, that avoids a monochromatic copy of K_{n_i} in color i, for all $i \in \{2, \ldots, t\}$. Replace each of the vertices in this critical coloring with a copy of $K_{c(G)-1}$ in color 1. The resulting t-coloring of

$$K_{(c(G)-1)(r(K_{n_2},\ldots,K_{n_t})-1)}$$

avoids monochromatic copies of K_{n_i} in color i, for all $i \in \{2, \ldots, t\}$, since such a complete graph would use at most a single vertex from each of the $K_{c(G)-1}$-subgraphs in color 1. It also avoids a copy of G in color 1 since the largest connected component in color 1 has order $c(G) - 1$. The desired inequality then follows. □

When equality holds for the inequality given in Lemma 2.1, we say that G is $(K_{n_2}, \dots, K_{n_t})$-*good*. In the following theorem, we show that F_m is (K_3, K_3)-good when $m \geq 6$.

Theorem 2.1. *For all* $m \geq 6$, $r(F_m, K_3, K_3) = 10m + 1$.

Proof. The lower bound $r(F_m, K_3, K_3) \geq 10m + 1$ follows from Lemma 2.1 and holds for all $m \geq 2$. To prove the reverse inequality, consider a 3-coloring of K_{10m+1}, where $m \geq 6$, using the colors red, blue, and green. If we group the colors blue and green together, then the evaluation $r(F_m, K_6) = 10m + 1$ proved in [13] implies that there exists a red F_m or a blue and green K_6. In the latter case, $r(K_3, K_3) = 6$ [8] implies that there is a blue K_3 or a green K_3. □

For any graph G, let $\delta(G)$ denote the *minimum degree* of G:

$$\delta(G) := \min\{\deg(x) \mid x \in V(G)\}.$$

The following lemma then gives a new lower bound for certain star-critical Ramsey numbers.

Lemma 2.2. *Let* $t \geq 3$ *and assume that* G *is* $(K_{n_2}, \dots, K_{n_t})$*-good. Then*

$$r_*(G, K_{n_2}, \dots, K_{n_t}) \geq (c(G) - 1)(r(K_{n_2}, \dots, K_{n_t}) - 2) + \delta(G).$$

Proof. Since G is $(K_{n_2}, \dots, K_{n_t})$-good, it follows that

$$r(G, K_{n_2}, \dots, K_{n_t}) = (c(G) - 1)(r(K_{n_2}, \dots, K_{n_t}) - 1) + 1.$$

Start with a $(t-1)$-coloring of

$$K_{r(K_{n_2},\dots,K_{n_t})-1} \sqcup K_{1,r(K_{n_2},\dots,K_{n_t})-2} = K_{r(K_{n_2},\dots,K_{n_t})} - e$$

that avoids a monochromatic copy of K_{n_i} in color i, for all $i \in \{2, \dots, t\}$. Such a coloring exists by Theorem 1.1 of [1]. Let $e = uv$ be the missing edge. Now replace each vertex, except for v, with copies of $K_{c(G)-1}$ in color 1, coloring all edges joining distinct blocks with the color of the edge that originally joined the vertices that were replaced. Edges that join v to other $K_{c(G)-1}$-blocks receive the same color as the edge that originally joined v to the vertex that was replaced.

Let x be a vertex in the copy of $K_{c(G)-1}$ that replaced vertex u. For every vertex y other than v or x, color edge vy the same color as edge vx. The resulting

$$K_{(c(G)-1)(r(K_{n_2},\dots,K_{n_t})-1)} \sqcup K_{1,(g(G)-1)(r(K_{n_2},\dots,K_{n_t})-2)}$$

still avoids a copy of K_{n_i} in color i, for all $i \in \{2, \dots, t\}$, and it avoids a copy of G in color 1. By joining $\delta(G) - 1$ edges in color 1 from v to the $K_{c(G)-1}$ that replaced u, we avoid a monochromatic copy of G in color 1 since vertex v cannot be included in such a copy of G. It follows that

$$r_*(G, K_{n_2}, \dots, K_{n_t}) > (c(G) - 1)(r(K_{n_2}, \dots, K_{n_t}) - 2) + \delta(G) - 1,$$

completing the proof. □

By Theorem 2.1, F_m is (K_3, K_3)-good when $m \geq 6$, so Lemma 2.2 implies the following theorem.

Theorem 2.2. *For all* $m \geq 6$, $r_*(F_m, K_3, K_3) \geq 8m + 2$.

Unfortunately, the proof given for the upper bound in Theorem 2.1 does not hold for values of m such that $2 \leq m \leq 5$. In order to make some additional progress, we now transition to Gallai-Ramsey theory.

3. Gallai-Ramsey theory for a fan versus complete graphs

We start by noting that the construction given in the proof of Lemma 2.1 holds for Gallai t-colorings. So, the statement

$$gr(G, K_{n_2}, \dots, K_{n_t}) \geq (c(G) - 1)(gr(K_{n_2}, \dots, K_{n_t}) - 1) + 1 \quad (1)$$

is also true. When equality holds for the inequality given in (1), we say that G is *Gallai* $(K_{n_2}, \dots, K_{n_t})$*-good*. In 1983, Chung and Graham [6] proved that for all $t \geq 2$,

$$gr(\underbrace{K_3, \dots, K_3}_{t \text{ terms}}) = \begin{cases} 5^{t/2} + 1 & \text{if } t \text{ is even} \\ 2 \cdot 5^{(t-1)/2} + 1 & \text{if } t \text{ is odd.} \end{cases}$$

Combining this result with (1), it follows that

$$gr(F_m, \underbrace{K_3, \ldots, K_3}_{t \text{ terms}}) \geq \begin{cases} 2m \cdot 5^{t/2} + 1 & \text{if } t \text{ is even} \\ 4m \cdot 5^{(t-1)/2} + 1 & \text{if } t \text{ is odd,} \end{cases} \tag{2}$$

for all $m, t \geq 2$.

Now we focus on the case of F_2 versus two copies of K_3, first giving a useful property of what will turn out to be the critical colorings for $gr(F_2, K_3, K_3)$. We require the following result of Liu, Magnant, Saito, Schiermeyer, and Shi [16] (see also Theorem 3.20 of [17]):

$$gr(K_4, K_3, K_3) = 17. \tag{3}$$

Lemma 3.1. *Every Gallai 3-coloring of K_{20} that avoids a red F_2, a blue K_3, and a green K_3 contains a red $5K_4$.*

Proof. Consider a Gallai 3-coloring of K_{20} (using red, blue, and green) that avoids a red F_2, a blue K_3, and a green K_3. Denote its vertex set by V. By Equation (3), there must exist a red K_4, and we denote its vertex set by V_1 and select a vertex $x_1 \in V_1$. Now consider the Gallai 3-coloring of K_{19} induced by the vertex set $V - \{x_1\}$. Applying (3) again, there exists a red K_4, whose vertex set we denote by V_2. Note that that $|V_1 \cap V_2| \in \{0, 1, 2, 3\}$. However, if any of the cases $|V_1 \cap V_2| \in \{1, 2, 3\}$ hold, then a red F_2 is formed (see Figure 3.1). So, $V_1 \cap V_2 = \emptyset$. Select a vertex $x_2 \in V_2$. Repeating this process, the K_{18}

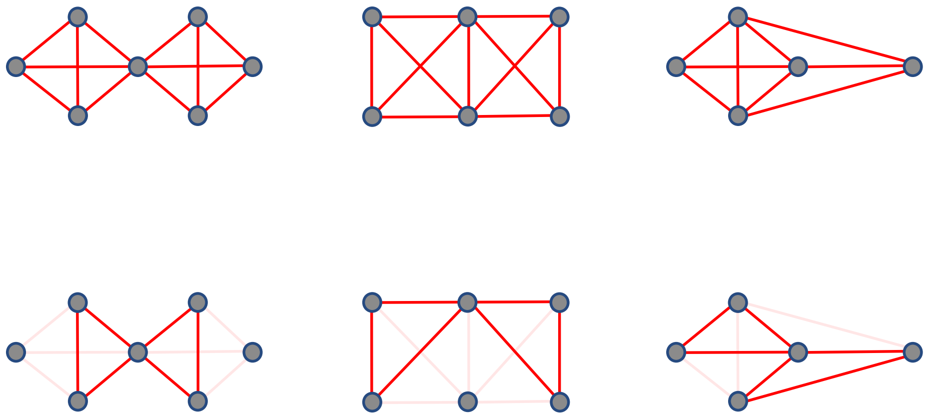

Figure 3.1: Three cases where two red K_4-subgraphs share 1, 2, and 3 vertices, respectively. In all three cases, a red F_2 is a subgraph, as is highlighted in the second row.

induced by the vertex set $V - \{x_1, x_2\}$ contains a red K_4, whose vertex set V_3 must be disjoint from $V_1 \cup V_2$ by the previous argument. Pick a vertex $x_3 \in V_3$. Consider the K_{17} induced by $V - \{x_1, x_2, x_3\}$, which again contains a red K_4 that is necessarily disjoint from $V_1 \cup V_2 \cup V_3$. Denote its vertex set by V_4. The vertex sets V_1, V_2, V_3, V_4 are pairwise disjoint and each induces a red K_4. So, our coloring contains a red $4K_4$.

Observe that the edges joining a pair of vertex sets V_i and V_j ($i \neq j$), cannot include both of the colors blue and green. To see this, suppose that $V_i = \{a, b, c, d\}$ and $V_j = \{w, x, y, z\}$. Without loss of generality, suppose that aw is blue. Then, since a red F_2 is avoided, a joins to at most one of $\{x, y, z\}$ with a red edge. If a rainbow K_3 is to be avoided, then a must join to at least two of the elements in $\{x, y, z\}$ via blue edges. Assume that ax and ay are blue. If any green edge joins the two red K_4-subgraphs, then it must be incident with z, and it cannot be az. Without loss of generality, suppose that bz is green (see the first image in Figure 3.2). Avoiding a rainbow K_3, edges bx and by must be red (see the second image in Figure 3.2), but then $\{b\} \cup V_j$ contains a red F_2. It follows that besides a potential matching of red edges joining V_i and V_j, all other edges must be the same color (either blue or green, but not both). We refer to such a color as the *dominant color* joining V_i and V_j and note that the dominant color appears on at least 12 edges joining V_i and V_j.

Let V_5 consist of the four vertices not contained in $V_1 \cup V_2 \cup V_3 \cup V_4$. If $u \in V_5$, then u can join to each V_i, where $i \in \{1, 2, 3, 4\}$ using at most a single red edge and with blue or green edges, but not both. Once again, we can talk about the dominant color of the edges joining u to V_i and note that the dominant color appears on at least 3 of the edges joining u and V_i. We conclude the proof by considering two cases.

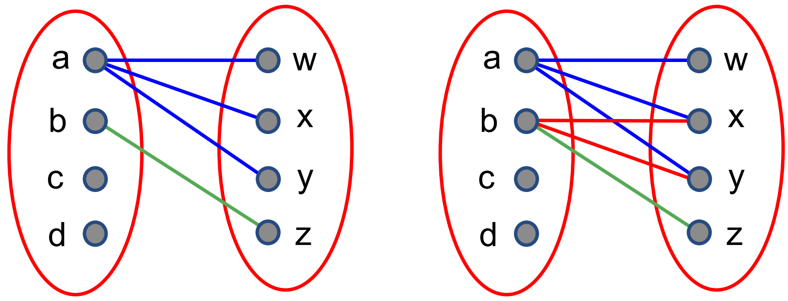

Figure 3.2: Joining edges in two disjoint red K_4-subgraphs in a Gallai 3-coloring of K_{20}.

Case 1. Suppose that some $u \in V_5$ joins to at least three of V_1, V_2, V_3, and V_4 using the same dominant color. Without loss of generality suppose that blue is the dominant color joining u to V_1, V_2, and V_3. If the dominant color joining V_1 and V_2 is blue, then a blue K_3 can be formed by including u and a single vertex from each of V_1 and V_2. For example, if ua, ub, and uc are blue with $a \in V_1$ and $b, c \in V_2$ (b and c are assumed to be distinct), then at most one of ab and ac is red and the other is blue, forming a blue K_3 with u. A similar argument follows if the dominant color is blue for the edges joining V_2 and V_3 or the edges joining V_1 and V_3. So, the dominant color for edges joining all three of the vertex sets V_1, V_2, and V_3 is green, and a green K_3 can be formed. As we obtain a blue K_3 or a green K_3, this case cannot occur.

Case 2. Suppose that each vertex in V_5 joins to exactly two of V_1, V_2, V_3, and V_4 with a dominant color blue and two with a dominant color green. Assume that some edge in V_5 (say, uv) is blue. Without loss of generality, suppose that blue is the dominant color joining u to V_1 and V_2 and green is the dominant color joining u to V_3 and V_4. If v joins with dominant color blue to either V_1 or V_2, then a blue K_3 can be formed. So, v must join with dominant color blue to V_3 and V_4 and with dominant color green to V_1 and V_2. Regardless of which dominant color (blue or green) appears joining V_1 to V_2, that color forms a blue or green K_3 with either u or v. This entire argument can be repeated if uv is green. Thus, it follows that uv must be red. Since u and v were arbitrary vertices in V_5, it follows that V_5 is a red K_4, and hence, our coloring of K_{20} contains a red $5K_4$. □

Theorem 3.1. *In the case of F_2 versus two copies of K_3, we have*

$$gr(F_2, K_3, K_3) = 21 \quad \textit{and} \quad gr_*(F_2, K_3, K_3) = 18.$$

Proof. The lower bound $gr(F_2, K_3, K_3) \geq 21$ follows from (2). We now prove that

$$gr(F_2, K_3, K_3) \leq 21 \quad \text{and} \quad gr_*(F_2, K_3, K_3) \leq 18$$

by showing that every Gallai 3-coloring of $K_{20} \sqcup K_{1,18}$ contains a red F_2, a blue K_3, or a green K_3. Consider such a coloring and let v be the vertex with degree 18. If the K_{20} avoids a red F_2, and blue K_3, and a green K_3, then by Lemma 3.1, it must contain a red $5K_4$, whose vertex sets we label V_1, V_2, V_3, V_4, and V_5. As was noted in the proof of Lemma 3.1, between each distinct pair of vertex sets V_i and V_j, there is a dominant color (either blue or green) that appears on all edges, except possibly a red matching. In order for a blue K_3 and a green K_3 to be avoided, the dominant colors for the edges spanning V_1, V_2, V_3, V_4, and V_5 must form a blue C_5 and a green C_5 (see Figure 3.3) as this coloring corresponds with the only critical coloring for $r(K_3, K_3)$.

Note that v must join to each V_i using at most one red edge and with all other edges either blue or green (but not both). So, as before, we can talk about a dominant color (blue or green) joining v to each V_i. Since 18 edges join v to the K_{20}, each V_i joins to v with at least two edges, one of which must be a color other than red. By the Pigeonhole Principle, v must join to at least three of V_1, V_2, V_3, V_4, and V_5 with the same dominant color. Without loss of generality, suppose that v has dominant color blue joining to V_1, V_2, and V_3. Then if any pair of these vertex sets join with dominant color blue, a blue K_3 can be formed. Otherwise, all three of them must join with dominant color green, and a green K_3 can be formed. It follows that

$$gr(F_2, K_3, K_3) \leq 21 \quad \text{and} \quad gr_*(F_2, K_3, K_3) \leq 18.$$

To complete the proof of the theorem, it remains to be shown that

$$gr_*(F_2, K_3, K_3) \geq 18.$$

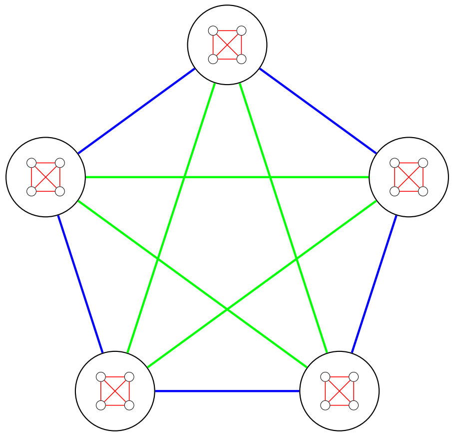

Figure 3.3: A Gallai 3-coloring of K_{20} that avoids a red F_2, a blue K_3, and a green K_3.

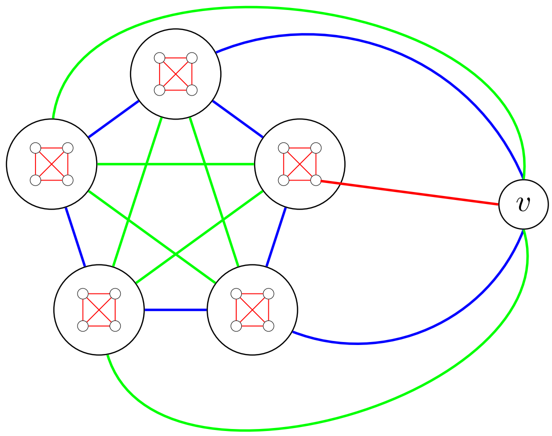

Figure 3.4: A Gallai 3-coloring of $K_{20} \sqcup K_{1,17}$ that avoids a red F_2, a blue K_3, and a green K_3.

Consider a critical coloring for $r(K_3, K_3)$ in blue and green (the base graph) and replace each of its vertices with red K_4-subgraphs (again, see Figure 3.3). Label the K_4 vertex sets V_1, V_2, V_3, V_4, and V_5 and suppose that $V_1V_2V_3V_4V_5V_1$ corresponds with a blue C_5 in the base graph. Introduce vertex v, joining it to V_1 and V_4 with blue edges, to V_2 and V_3 with green edges, and to a single vertex in V_5 with a red edge (see Figure 3.4). The resulting $K_{20} \sqcup K_{1,17}$ avoids a blue K_3 and a green K_3. Vertex v cannot be included in a red F_2 since it has red degree 1 and no red connected component in the K_{20} is large enough to contain a red F_2. Thus, $gr_*(F_2, K_3, K_3) \geq 18$, completing the proof of the theorem. □

4. Conclusion

We now conclude by stating some conjectures and directions for future research motivated by our work. The following conjecture generalizes Conjecture 1.1.

Conjecture 4.1. *For all* $t \geq 3$ *and* $m \geq gr(K_{n_2}, \ldots, K_{n_t}) \geq 3$,

$$gr(F_m, K_{n_2}, \ldots, K_{n_t}) = 2m(gr(K_{n_2}, \ldots, K_{n_t}) - 1) + 1.$$

Whenever Conjecture 1.1 is true, we obtain the following conjecture concerning the corresponding star-critical Ramsey numbers.

Conjecture 4.2. *If* $r(F_m, K_n) = 2m(n-1) + 1$ *for some fixed* $m \geq n \geq 3$, *then*

$$r_*(F_m, K_n) = 2m(n-2) + 2.$$

Of course, this conjecture can be further generalized to the multicolor case.

Conjecture 4.3. *If* $gr(F_m, K_{n_2}, \ldots, K_{n_t}) = 2m(gr(K_{n_2}, \ldots, K_{n_t}) - 1) + 1$ *for some* $t \geq 3$ *and* $m \geq gr(K_{n_2}, \ldots, K_{n_t}) \geq 3$*, then*

$$gr_*(F_m, K_{n_2}, \ldots, K_{n_t}) = 2m(gr(K_{n_2}, \ldots, K_{n_t}) - 2) + 2.$$

Other related variations of Ramsey numbers that have been considered include $r(K_1 + mK_3, K_n)$ [11], $r(K_1 + mK_t, K_n)$ [19], and $r(K_1 + mH, kK_n)$ [10], along with the corresponding star-critical Ramsey numbers. At present, no multicolor analogues for these numbers have been studied.

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