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Research Article

On unicyclic graphs with a given girth and their minimum symmetric division deg index

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Abstract

The degree of a vertex u in a graph G is denoted by $d_G(u)$. The symmetric division deg (SDD) index of G is denoted by $SDD(G)$ and is defined as $SDD(G) = \sum_{xy\in E(G)}[d_G(x)/d_G(y) + d_G(y)/d_G(x)]$, where $E(G)$ is the set of all edges of G. A connected graph with the same number of vertices and edges is known as a unicyclic graph. The girth of a unicyclic graph is the number of edges of its unique cycle. This paper solves the problem of characterizing graphs attaining the first two minimum values of the SDD index over the class of all unicyclic graphs of fixed order and with a given girth. Applications of the obtained results yield the solution to the problem of determining graphs having the first three minimum values of the SDD index among all unicyclic graphs of a given order.

Keywords: topological index; symmetric division deg index; unicyclic graph; girth of a graph.

2020 Mathematics Subject Classification: 05C07, 05C09, 05C35.

1. Introduction

The graphs considered in this paper are connected and finite. The graph-theoretical terms used in this paper, but not defined here, can be found in some standard books, like [6, 7, 14].

By a graph invariant, we mean a property of graphs that is preserved by graph isomorphism. We remark here that a graph invariant may be the same for two non-isomorphic graphs. In chemical graph theory, the graph invariants that take only real numbers are often referred to as topological indices [21,24] or molecular (structure) descriptors. Many topological indices have chemical application; for example, see [12,13,20]. The symmetric division deg (SDD) index, introduced in [23], is a topological index that has a strong correlation with the total surface area of polychlorobiphenyls [23]. This topological index, for a graph G, is defined as

$$SDD(G) = \sum_{xy\in E(G)} \left(\frac{d_G(x)}{d_G(y)} + \frac{d_G(y)}{d_G(x)}\right), \tag{1}$$

where $E(G)$ is the edge set of G and $d_G(x)$ denotes the degree of the vertex $x \in V(G)$. The ratios of the arithmetic and harmonic means of the degrees of the end-vertices of edges of G may also be used to produce a topological index involving the SDD index (see [2]). Also, the SDD index of G can be rewritten [4] as

$$SDD(G) = 2|E(G)| + \sum_{xy\in E(G)} \frac{(d_G(x) - d_G(x))^2}{d_G(x)d_G(y)},$$

which may be preferred over the formula (1) in certain circumstances. Furtula et al. [10] compared the applicability of the SDD index with certain popular topological indices and found it as a viable topological index, surpassing the predictive performance of several indices. Vasilyev [22] appears to have started the comprehensive study of mathematical aspects related to the SDD index. Most of the known bounds and extremal results related to the SDD index can be found in the survey paper [5]. Das [8] solved one of the open problems, concerning the SDD index, posed in [5]. In the recent paper [3], an upper bound on the SDD index of a graph in terms of its order, size and maximum degree is derived. The reader may also consult the references [1, 9, 11, 15, 16, 18, 19] for additional detail about the SDD index.

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A (connected) graph with the same number of vertices and edges is known as a unicyclic graph. The girth of a unicyclic graph is the number of edges of its unique cycle. The present paper solves the problem of characterizing graphs attaining the first two minimum values of the SDD index over the class of all unicyclic graphs of fixed order and with a given girth. Applications of the obtained results yield the solution to the problem of determining graphs having the first three minimum values of the SDD index among all unicyclic graphs of a given order. The latter problem was solved partially by Vasilyev [22] and completely by Pan and Li [17].

2. Preliminaries

In this section, we define some notations and terms used in the subsequent section.

A graph of order 1 is known as a trivial graph. By an n-order graph, we mean a graph with order n. The n-order path graph and cycle graph are denoted by P_n and C_n, respectively. For a vertex x of a graph G, we use the notation $N_G(x)$ to represent the set of all those vertices of G that are adjacent to x. The members of $N_G(x)$ are called the neighbors of x. A vertex x of a graph G with $d_G(x) = 1$ is known as a pendent vertex. A non-trivial path $P : x_1x_2 \cdots x_p$ in a graph G is said to be a pendent path of G if

$$\min\{d_G(x_1), d_G(x_p)\} = 1, \quad \max\{d_G(x_1), d_G(x_p)\} \geq 3, \quad \text{and} \quad d_G(x_i) = 2 \quad \text{when } 2 \leq i \leq p-1.$$

We end this section with the following known result, which is used in the proofs of the main results of this paper:

Lemma 2.1 (see [17])**.** *For a graph G having m edges and r pendent paths, the following inequality holds:*

$$SDD(G) \geq 2m + \frac{2r}{3}.$$

3. Results

Denote by $\mathscr{U}_{n,k}$ the set of all n-order unicyclic graphs with girth k, where $3 \leq k \leq n$. Certainly, set class $\cup_{k=3}^{n}\mathscr{U}_{n,k}$ consists of all n-order unicyclic graphs. Note that the class $\mathscr{U}_{n,n}$ consists of only the cycle graph C_n. Also, observe that the class $\mathscr{U}_{n,n-1}$ consists of only one graph; namely, the graph obtained from the cycle graph C_{n-1} by attaching exactly one pendent vertex to any vertex of C_{n-1}. For $3 \leq k \leq n-2$, denote by $\mathfrak{U}_{n,k}$ the n-order graph created from the k-order cycle C_k and $(n-k)$-order path P_{n-k} by inserting an edge between a pendent vertex of P_{n-k} and a vertex of C_k. The graph $\mathfrak{U}_{n,k}$ is shown in Figure 3.1.

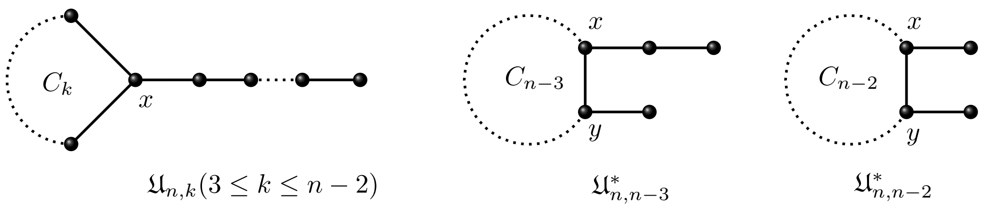

Figure 3.1: The graph $\mathfrak{U}_{n,k}(3 \leq k \leq n-2)$ and the unique graphs belonging to the classes $\mathfrak{U}^*_{n,n-3}$ and $\mathfrak{U}^*_{n,n-2}$.

Theorem 3.1. *If $G \in \mathscr{U}_{n,k}$ with $3 \leq k \leq n-2$, then*

$$SDD(G) \geq 2n + 1,$$

where the equality holds if and only if $G \cong \mathfrak{U}_{n,k}$ (see Figure 3.1). Equivalently, the graph $\mathfrak{U}_{n,k}$ uniquely attains the minimum SDD index, which is $2n+1$, over the class $\mathscr{U}_{n,k}$ for $3 \leq k \leq n-2$.

Proof. Let r be the number of pendent paths of G. If $r = 1$, then $G \cong \mathfrak{U}_{n,k}$ and hence

$$SDD(G) = 2n + 1.$$

If $r \geq 2$, then Lemma 2.1 implies that

$$SDD(G) \geq 2n + \frac{2r}{3} \geq 2n + \frac{2(2)}{3} > 2n + 1,$$

which completes the proof. □

Since $SDD(C_n) = 2n$ and the SDD index of the n-order unicyclic graph with girth $n-1$ is $2n + \frac{5}{3}$, from Theorem 3.1 the next two results follow; particularly, the first one of these two results was first established in [22] and the second one was derived in [17].

Corollary 3.1. *For every integer n greater than 4, the cycle graph C_n uniquely attains the minimum SDD index (which is $2n$) over the class of all n-order unicyclic graphs. Equivalently, if G is an n-order unicyclic graph such that $n \geq 5$, then*

$$SDD(G) \geq 2n,$$

with equality if and only if $G \cong C_n$.

Corollary 3.2. *For every integer n greater than 4, only the graph(s) of the class $\{\mathfrak{U}_{n,k} : 3 \leq k \leq n-2\}$ attain(s) the second-minimum value of the SDD index among all n-order unicyclic graphs, where the graph $\mathfrak{U}_{n,k}$ is depicted in Figure 3.1. (The mentioned second-minimum value of the SDD index is $2n+1$.) Equivalently, if G is an n-order unicyclic graph different from the cycle C_n such that $n \geq 5$, then*

$$SDD(G) \geq 2n + 1,$$

with equality if and only if $G \in \{\mathfrak{U}_{n,k} : 3 \leq k \leq n-2\}$.

Next, we find the second-minimum value of the SDD index of the graphs belonging to the class $\mathscr{U}_{n,k}$. For this, we first define certain graphs. Let x be the unique vertex of degree 3 in the graph $\mathfrak{U}_{n,k}$ (see Figure 3.1). When $k \in \{n-3, n-2\}$, we denote by $\mathfrak{U}^*_{n,k}$ the class of the unique graph obtained from $\mathfrak{U}_{n,k}$ by removing its unique pendent vertex and making it adjacent with one (say y) of those two neighbors of x that lie on the cycle. The graphs $\mathfrak{U}^*_{n,n-3}$ and $\mathfrak{U}^*_{n,n-2}$ are also depicted in Figure 3.1.

For $3 \leq k \leq n-4$, denote by $\mathfrak{U}^*_{n,k}$ the class of the graph(s) obtained from the cycle C_k by attaching two pendent paths of lengths ℓ_1 and ℓ_2 at the vertices $x, y \in V(C_k)$ (one at x and the other at y), where $\min\{\ell_1, \ell_2\} \geq 2$ and $xy \in E(C_k)$. For $3 \leq k \leq n-5$, denote by $\mathfrak{U}^\dagger_{n,k}$ the class of the graph(s) obtained from the cycle C_k and the path P_{n-k} by inserting an edge between a vertex x of C_k and a vertex y of P_{n-k}, where the distance between y and each of the pendent vertices of P_{n-k} is at least 2. The general forms of the graphs belonging to $\mathfrak{U}^*_{n,k}$ and $\mathfrak{U}^\dagger_{n,k}$ are shown in Figure 3.2.

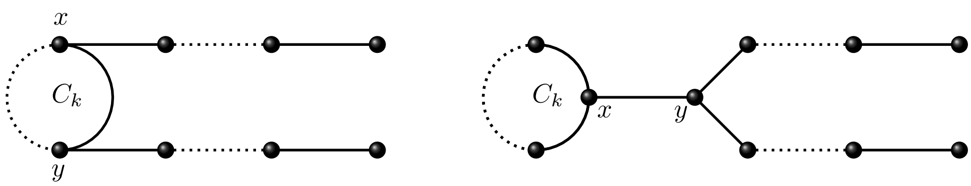

Figure 3.2: The general forms of the graphs belonging to $\mathfrak{U}^*_{n,k}$ (left) and $\mathfrak{U}^\dagger_{n,k}$ (right). The length of every pendent path in each depicted graph is at least 2.

Theorem 3.2. *Let $G \in \mathscr{U}_{n,k}$ such that $G \ncong \mathfrak{U}_{n,k}$ (see Figure 3.1).*

(i). *If $3 \leq k \leq n-4$, then*

$$SDD(G) \geq 2n + \frac{5}{3}\,. \tag{2}$$

If $3 \leq k \leq n-5$, then the equality in (2) holds if and only if either $G \in \mathfrak{U}^*_{n,k}$ or $G \in \mathfrak{U}^\dagger_{n,k}$ (see Figure 3.2). If $k = n-4$, then the equality in (2) holds if and only if $G \in \mathfrak{U}^*_{n,k}$.

(ii). *If $k = n-3$, then*

$$SDD(G) \geq 2n + \frac{7}{3}\,,$$

where the equality holds if and only if $G \in \mathfrak{U}^*_{n,n-3}$ (see Figure 3.1).

(iii). *If $k = n-2$, then*

$$SDD(G) \geq 2n + 3\,,$$

where the equality holds if and only if $G \in \mathfrak{U}^*_{n,n-2}$ (see Figure 3.1).

Proof. Let r be the number of pendent paths of G. Since $G \not\cong \mathfrak{U}_{n,k}$, it holds that $r \geq 2$.

(i). If $r \geq 3$, then Lemma 2.1 implies that

$$SDD(G) \geq 2n + \frac{2r}{3} \geq 2n + \frac{2(3)}{3} > 2n + \frac{5}{3}.$$

Now, assume that $r = 2$. Then, the graph G belongs to one of the three classes of graphs as shown in Figure 3.3; that is, $G \in \mathfrak{U}^1_{n,k} \cup \mathfrak{U}^2_{n,k} \cup \mathfrak{U}^3_{n,k}$. These classes are defined as follows. The class $\mathfrak{U}^1_{n,k}$ consists of the graph(s) obtained from the k-order cycle C_k by attaching two pendent paths at two distinct vertices $x, y \in V(C_k)$ (one at x and the other at y) such that one of the pendent paths has length a and the other has length b, where $a \geq b \geq 1$ and $a + b = n - k$. The class $\mathfrak{U}^2_{n,k}$ consists of the graph(s) obtained from the k-order cycle C_k by attaching two pendent paths at one vertex $x \in V(C_k)$ such that one of the pendent paths has length a and the other has length b, where $a \geq b \geq 1$ and $a + b = n - k$. The class $\mathfrak{U}^3_{n,k}$ consists of the graph(s) obtained from the k-order cycle graph C_k and the $(a+1)$-order path graph P_{a+1} by inserting a path of length b between a vertex $x \in V(C_k)$ and a non-pendent vertex $y \in V(P_{a+1})$, where $a \geq 2$, $b \geq 1$ and $a + b = n - k$.

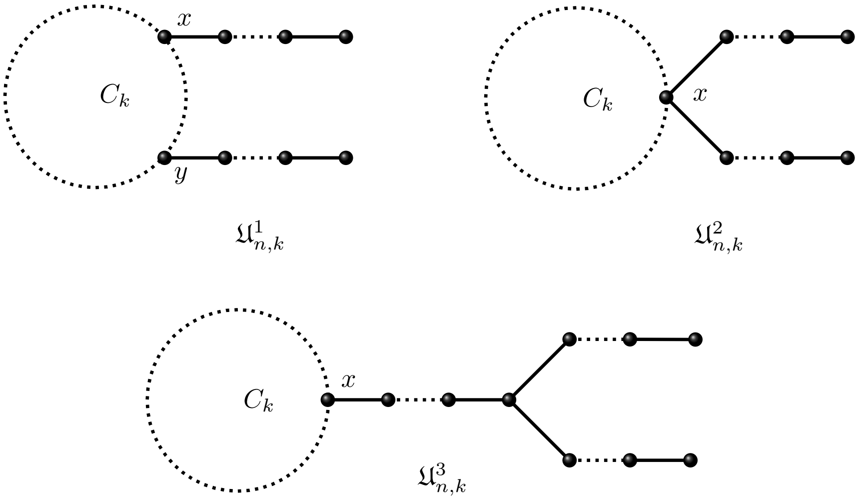

Figure 3.3: The general forms of the graphs belonging to the classes $\mathfrak{U}^1_{n,k}$, $\mathfrak{U}^2_{n,k}$, and $\mathfrak{U}^3_{n,k}$, which are used in the proof of Theorem 3.2(i), where $3 \leq k \leq n - 4$.

Case 1. $G \in \mathfrak{U}^1_{n,k}$.
Recall that $a \geq b \geq 1$ and $a + b = n - k$. Since $k \leq n - 4$, the integers a and b cannot be simultaneously equal to 1. If $xy \in E(G)$, then

$$SDD(G) = \begin{cases} 2n + \dfrac{7}{3} & \text{when } a > b = 1, \\[2ex] 2n + \dfrac{5}{3} & \text{when } a \geqslant b > 1. \end{cases}$$

If $xy \notin E(G)$, then

$$SDD(G) = \begin{cases} 2n + \dfrac{8}{3} & \text{when } a > b = 1, \\[2ex] 2(n+1) & \text{when } a \geqslant b > 1. \end{cases}$$

Case 2. $G \in \mathfrak{U}^2_{n,k}$.
Note that, in this case too, the integers a and b cannot be simultaneously equal to 1. Hence, we have

$$SDD(G) = \begin{cases} 2n + \dfrac{17}{4} & \text{when } a > b = 1, \\[2ex] 2n + 3 & \text{when } a \geqslant b > 1. \end{cases}$$

Case 3. $G \in \mathfrak{U}^3_{n,k}$.
In this case, recall that $a \geq 2$, $b \geq 1$, and $a + b = n - k$.

First, assume that $b = 1$. Since $k \leq n-4$, we have $a \geq 3$ and hence the vertex y (that is the common vertex of the pendent paths of $\mathfrak{U}^3_{n,k}$) is adjacent to at most one pendent vertex. Thus, we have

$$SDD(G) = \begin{cases} 2n + \dfrac{7}{3} & \text{when the vertex } y \text{ is adjacent to exactly one pendent vertex,} \\ 2n + \dfrac{5}{3} & \text{when the vertex } y \text{ is not adjacent to any pendent vertex.} \end{cases}$$

Next, we assume that $b > 1$. Then, we have

$$SDD(G) = \begin{cases} 2n + \dfrac{10}{3} & \text{when the vertex } y \text{ is adjacent to two pendent vertices,} \\ 2n + \dfrac{8}{3} & \text{when the vertex } y \text{ is adjacent to exactly one pendent vertex,} \\ 2(n+1) & \text{when the vertex } y \text{ is not adjacent to any pendent vertex.} \end{cases}$$

By combining the values of the SDD index obtained in all the above cases and after an elementary comparison, we arrive at the desired conclusion of part **(i)**.

(ii). Since $k = n-3$, we have $r \in \{2, 3\}$.

Case 1. $r = 3$.
In the considered case, the graph G belongs to one of the three classes of graphs whose general forms are shown in Figure 3.4; that is, $G \in H_1 \cup H_2 \cup H_3$.

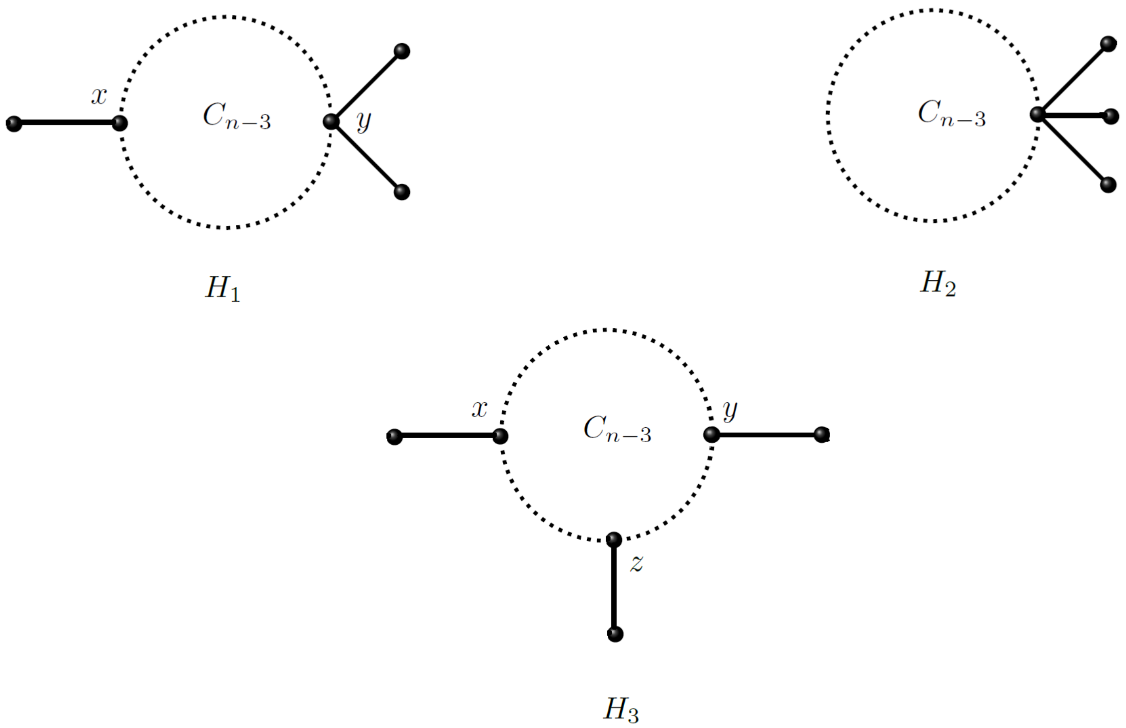

Figure 3.4: The general forms of the graphs belonging to the classes H_1, H_2, and H_3, which are used in the proof of Theorem 3.2(ii).

Subcase 1.1. $G \in H_1$.
If there is an edge between x and y, then

$$SDD(G) = 2n + \frac{79}{12}.$$

If x and y are not adjacent, then

$$SDD(G) = 2n + \frac{43}{6}.$$

Subcase 1.2. $G \in H_2$.
Note that the class H_2 consists of only one graph and hence we have

$$SDD(G) = 2n + \frac{57}{5}\,.$$

Subcase 1.3. $G \in H_3$.
If all the three vertices x, y, and z are pairwise adjacent, then $n = 6$ and hence

$$SDD(G) = 16 > 2n + \frac{7}{3}\,.$$

If there is no edge between the vertices of exactly one of the pairs (x, y), (x, z), (y, z), then we have

$$SDD(G) = 2n + \frac{13}{3}\,.$$

If there is an edge between the vertices of exactly one of the pairs (x, y), (x, z), (y, z), then we have

$$SDD(G) = 2n + \frac{14}{3}\,.$$

If all the three vertices x, y, and z are pairwise non-adjacent, then we have $SDD(G) = 2n + 5$.

By comparing the obtained values of the SDD index in the above three subcases, we conclude that

$$SDD(G) > 2n + \frac{7}{3}\,,$$

as desired.

Case 2. $r = 2$.
In this case, either $G \in \mathfrak{U}^*_{n,n-3}$ or the graph G is one of the three graphs shown in Figure 3.5.

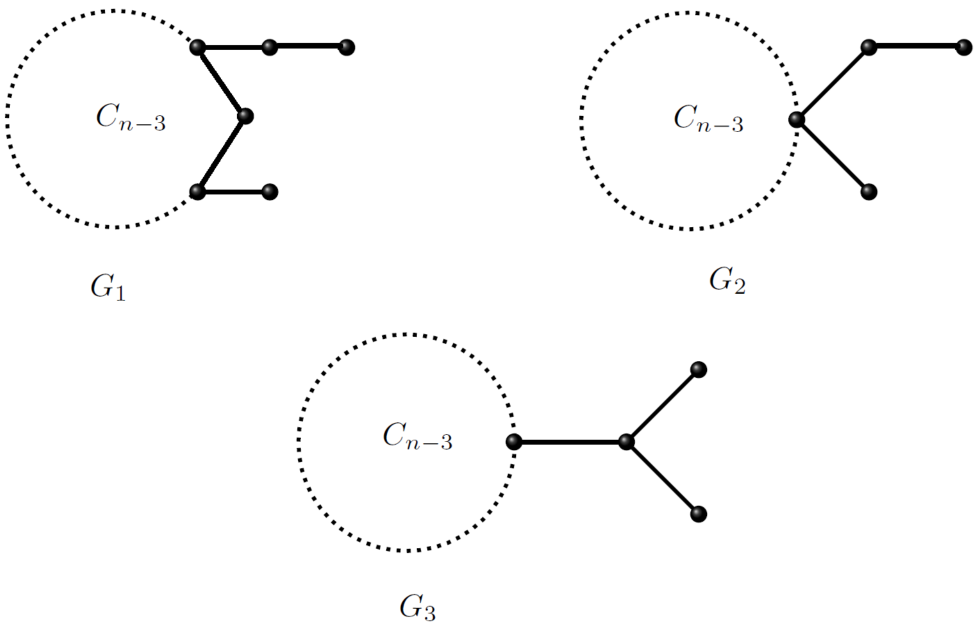

Figure 3.5: The graphs G_1, G_2, and G_3 used in the proof of Theorem 3.2(ii).

On the other hand, we have

$$SDD(G_1) = 2n + \frac{8}{3}\,,$$

$$SDD(G_2) = 2n + \frac{17}{4}\,,$$

$$SDD(G_3) = 2n + 3\,,$$

and

$$SDD(U) = 2n + \frac{7}{3},$$

where $\mathfrak{U}^*_{n,n-3} = \{U\}$. Consequently, we have

$$SDD(U) = 2n + \frac{7}{3} < SDD(G_i),$$

for every $i \in \{1, 2, 3\}$, as desired.

(iii). Note that the graph G belongs to either of the two classes of graphs, whose general forms are depicted in Figure 3.6.

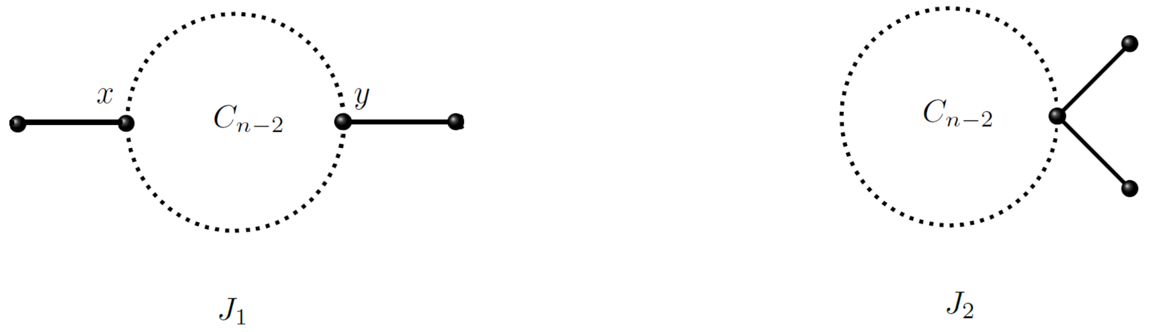

Figure 3.6: The general forms of the graphs belonging to the classes J_1 and J_2 used in the proof of Theorem 3.2(iii).

Case 1. $G \in J_1$.
If the vertices x and y are non-adjacent, then

$$SDD(G) = 2n + \frac{10}{3}.$$

If there is an edge between the vertices x and y, then $G \in \mathfrak{U}^*_{n,n-2}$ and $SDD(G) = 2n + 3$.

Case 2. $G \in J_2$.
In this case, we have

$$SDD(G) = 2n + \frac{11}{2}.$$

Hence, $SDD(G) \geq 2n + 3$ with equality if and only if $G \in \mathfrak{U}^*_{n,n-2}$. □

From Corollary 3.2 and Theorem 3.2, the next result follows, which was first established in [17].

Corollary 3.3. *For every integer n greater than 4, only the graph(s) of the class $H^* \cup (\cup_{k=3}^{n-4}\mathfrak{U}^*_{n,k}) \cup (\cup_{k=3}^{n-5}\mathfrak{U}^{\dagger}_{n,k})$ attain(s) the third-minimum value of the SDD index among all n-order unicyclic graphs, where H^* is the n-order graph of girth $n-1$ and the general forms of the graphs belonging to the classes $\mathfrak{U}^*_{n,k}$ and $\mathfrak{U}^{\dagger}_{n,k}$ (with $3 \leq k \leq n-4$) are depicted in Figure 3.2. (The mentioned third-minimum value of the SDD index is $2n + \frac{5}{3}$.) Equivalently, if G is an n-order unicyclic graph, not belonging to $\{C_n\} \cup \{\mathfrak{U}_{n,k} : 3 \leq k \leq n-2\}$, such that $n \geq 5$, then*

$$SDD(G) \geq 2n + \frac{5}{3},$$

with equality if and only if $G \in H^* \cup (\cup_{k=3}^{n-4}\mathfrak{U}^*_{n,k}) \cup (\cup_{k=3}^{n-5}\mathfrak{U}^{\dagger}_{n,k})$.

4. Concluding remarks

We have solved, in Theorems 3.1 and 3.2, the problem of characterizing graphs attaining the first two minimum values of the SDD index over the class of all unicyclic graphs of fixed order and with a given girth. By applying these theorems, in Corollaries 3.1, 3.2, and 3.3, we have rediscovered the solution to the problem of determining graphs having the first three minimum values of the SDD index over the class of all unicyclic graphs of a given order. The present study can be extended in several ways; for instance, it would be interesting to establish the maximal versions of Theorems 3.1 and 3.2. Also, in the survey paper [5], several open problems related to the SDD index were posed. It seems to be interesting to address those open problems; particularly, the one concerning the characterization of the graph(s) having the minimum SDD index over the class of all n-order unicyclic graphs with a given number of pendent vertices.

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