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Research Article

Coprime divisors graphs and their coloring parameters

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Abstract

Let n be a composite integer. The coprime divisors graph of n, denoted by Γ_n, is the graph whose vertices are the proper divisors of n and two vertices are adjacent if they are coprime. In this paper, we study the structure of Γ_n and its coloring parameters. Indeed, we give the explicit forms of the degrees, distances, diameter, and girth of this graph. We also compute explicitly the clique number, the chromatic number, and the independence number of Γ_n. Moreover, we prove that Γ_n is a perfect graph.

Keywords: divisors; perfect graph; chromatic number; independence number.

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1. Introduction

A graph, denoted by $G = (V, E)$, is a structure composed of two fundamental components: a non-empty set of objects called vertices, denoted by $V = \{v_1, v_2, \ldots\}$, and a multiset $E = \{e_1, e_2, \ldots\}$ whose elements are known as edges, where every e_i (if it exists) is a multiset consisting of two elements of V; namely, $e_i = \{v_j, v_k\}$. If V and E are finite, then G is called a finite graph. If $E = \emptyset$, then G is called a null graph. An edge $\{v_i, v_i\}$ in G is called a loop or self-loop. The elements of E having the multiplicity of at least 2 (if they exist) are known as multiple edges. The graph G is said to be simple if it contains neither a loop nor multiple edges.

Algebraic graph theory has become a very active research area in recent decades. It allows to create a connection between algebra and combinatorics. It provides a graphical representation of some algebraic structures and an algebraic modeling of some graphs. In this sense, Anderson and Livingston [3] introduced the zero-divisor graph to study zero-divisors of commutative rings using graphs. Later, several papers were published on the study of zero-divisor graphs; for example, see [1, 2, 7, 10, 11]. Recently, more algebraic structures were studied using graphs; for instance, we refer the reader to the socle of Artinian algebras studied by Neves [9], the clean graph introduced by Habibi et al. [5], the order divisor graphs of finite groups by Rehman et al. [13], and the cozero divisor graph studied by Rather [12].

In this paper, we introduce a new class of graphs associated with some arithmetical properties. Particularly, for any composite integer n, we denote by $D(n)$ the set of proper divisors of n and by Γ_n the coprime divisors graph whose set of vertices is $V(\Gamma_n) = D(n)$ and two vertices $x, y \in V(\Gamma_n)$ are adjacent if $gcd(x, y) = 1$. This work allows us to obtain a graphical representation of the relation between the proper divisors of n in the sense of the coprimeness. We compute explicitly the degrees of vertices, the girth, and the distances between connected vertices of the coprime divisors graph. We also determine explicitly some important coloring parameters of graphs; especially, the independence number, the chromatic number, and the clique number. Furthermore, we prove that Γ_n provides a new class of perfect graphs (see [4]).

2. Results

Let n be a positive integer. A positive divisor d of n is called a proper divisor of n if $1 < d < n$. We denote by $D(n)$ the set of all proper divisors of n.

Definition 2.1. *The coprime divisors graph of n, denoted by Γ_n, is the graph in which the vertices are the proper divisors of n, and two distinct vertices u and v are adjacent if and only if $gcd(u, v) = 1$.*

Definition 2.2. *Let n be a composite integer and $D(n)$ be its set of proper divisors. Then, the neighborhood set $V_n(x)$ of a vertex x in the graph Γ_n is defined as $\{y \in D(n) \mid gcd(x, y) = 1\}$.*

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Let us denote by $D_{pr}(n)$ the set of prime divisors of n, and $\nu(n) := |D_{pr}(n)|$.

Proposition 2.1. *Let n be a composite integer. Then, $D_{pr}(n)$ is a clique in Γ_n.*

Proof. For distinct prime divisors p and q in $D_{pr}(n)$, the graph Γ_n has an edge between them since $gcd(p, q) = 1$. Therefore, $D_{pr}(n)$ is a clique in Γ_n. □

Proposition 2.2. *Let n be a composite integer and $d \in D(n)$. There exists a prime integer $p \in D(n)$ such that p divides d.*

Proof. Since d is a divisor of n, any divisor of d is a divisor of n. On the other hand, d is a proper divisor. Then $d > 1$; namely, there is a prime integer p which divides d. Thus, $p \in D(n)$. □

Let p be a prime integer. Recall that the p-adic valuation of a nonzero integer n is defined as follows:

$$v_p(n) = \max\{k \in \mathbb{N} : p^k \text{ divides } n\}.$$

For more details on valuations, we refer the reader to [8]. Let us now compute the degrees of the vertices of Γ_n.

Theorem 2.1. *Let n be a composite integer and $d \in D(n)$. Then, the degree of d in the graph Γ_n is given as follows:*

- *If $v_p(d) \geq 1$, for any $p \in D_{pr}(n)$, then $deg_n(d) = 0$.*
- *Otherwise, $deg_n(d) = \prod_{p\in D_{pr}(n)} (v_p(n)\gamma_p(d) + 1) - 1$, where $\gamma_p(d) = \begin{cases} 0 & \text{if } v_p(d) \geq 1, \\ 1 & \text{if } v_p(d) = 0. \end{cases}$*

Proof. Let $d, d' \in D(n)$. Then, d is adjacent with d' if and only if d and d' are coprime. Equivalently, for any prime integer $p \in D_{pr}(n)$, if p divides d then p does not divide d'; namely, if $v_p(d) \geq 1$ then $v_p(d') = 0$. Let $D_{pr}(d) = \{p \in D_{pr}(n) \mid v_p(d) \geq 1\}$. Then, the neighborhood of d is obtained as follows:

$$V_n(d) = \{x \in D(n) \mid v_p(x) = 0, \ \forall p \in D_{pr}(d)\}.$$

- **Case 1:** if $V_n(d) = \emptyset$, then $v_p(d) \geq 1$, for any $p \in D_{pr}(n)$. In this case, we have $deg_n(d) = |V_n(d)| = 0$.
- **Case 2:** suppose that $V_n(d) \neq \emptyset$. Then, we have

$$V_n(d) = \left\{ \prod_{p\in D_{pr}(n)\setminus D_{pr}(d)} p^{r_p} \mid 0 \leq r_p \leq v_p(n) \right\} \setminus \{1\}.$$

  It follows that

$$deg_n(d) = |V_n(d)| = \prod_{p\in D_{pr}(n)\setminus D_{pr}(d)} (v_p(n) + 1) - 1.$$

  Let

$$\gamma_p(d) = \begin{cases} 0 & \text{if } v_p(d) \geq 1, \\ 1 & \text{if } v_p(d) = 0. \end{cases}$$

  Consequently, we have

$$deg_n(d) = \prod_{p\in D_{pr}(n)} (v_p(n).\gamma_p(d) + 1) - 1.$$

□

Example 2.1. *In the graph Γ_{30} shown in Figure 2.1, we have $n = 30 = 2 \times 3 \times 5$; namely, $v_2(30) = 1$, $v_3(30) = 1$, and $v_5(30) = 1$. Let $d = 3$. Then, $v_2(3) = 0$, $v_3(3) = 1$, and $v_5(3) = 0$. Therefore, $\gamma_2(3) = 1$, $\gamma_3(3) = 0$, and $\gamma_5(3) = 1$. It follows that*

$$deg_{30}(3) = 1 \times 2 \times 2 - 1 = 4 - 1 = 3.$$

Remark 2.1. *Let n be a composite integer and $Iso(n)$ the set of isolated vertices of Γ_n. Then, the size of Γ_n is given as follows:*

$$size(\Gamma_n) = \frac{1}{2} \sum_{d\in D(n)\setminus Iso(n)} deg_n(d).$$

Corollary 2.1. *Let n be a composite integer and $Iso(n)$ the set of isolated vertices in Γ_n. Then, the number of isolated vertices in Γ_n is $|Iso(n)| = \prod_{p\in D_{pr}(n)} v_p(n) - 1$.*

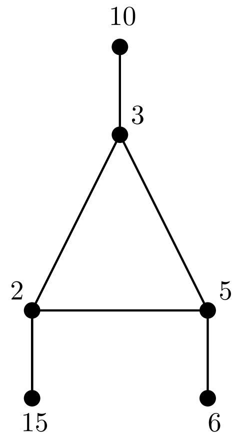

Figure 2.1: The graph Γ_{30}.

Proof. Let d be a vertex in Γ_n. Then, d is isolated if and only if $deg_n(d) = 0$. By Theorem 2.1, d is isolated if and only if $v_p(n) \geq 1$ for any $p \in D_{pr}(n)$. It follows that the set of isolated vertices can be written as follows:

$$Iso(n) = \left\{ \prod_{p \in D_{pr}(n)} p^{r_p} \mid 1 \leq r_p \leq v_p(n) \right\} \setminus \{1\}.$$

Thus, $|Iso(n)| = \prod_{p \in D_{pr}(n)} v_p(n) - 1$. □

Remark 2.2. *By the previous proof, we have that* $Iso(n) = \{\prod_{p \in D_{pr}(n)} p^{r_p} \mid 1 \leq r_p \leq v_p(n)\} \setminus \{1\}$. *Notice that* Γ_n *is connected if and only if* n *is squarefree.*

Theorem 2.2. *Let* n *be a composite integer. Let* x *and* y *be two non-isolated vertices of* Γ_n. *Then, the distance between* x *and* y *is given as follows:*

$$d(x, y) = \begin{cases} 1 & \text{if } gcd(x, y) = 1, \\ 2 & \text{if } gcd(x, y) > 1 \text{ and there exists } p \in D_{pr}(n) \text{ such that } p \text{ does not divide } xy, \\ 3 & \text{if } gcd(x, y) > 1 \text{ and } p \text{ divides } xy \text{ for any } p \in D_{pr}(n). \end{cases}$$

Proof. Let x and y be two non-isolated vertices in Γ_n.

- **Case 1:** suppose that $gcd(x, y) = 1$. Then, x and y are adjacent; namely, $d(x, y) = 1$.
- **Case 2:** suppose that $gcd(x, y) > 1$. Then, x and y are not adjacent; namely, $d(x, y) > 1$. We distinguish two subcases:
  - **Case 2.1:** suppose that there exists $p \in D_{pr}(n)$ such that p does not divide xy. Then, p divides neither x nor y. Since p is prime, we have $gcd(x, p) = 1$ and $gcd(y, p) = 1$; namely, x is adjacent with p and y is adjacent with p. Thus, $d(x, y) = 2$.
  - **Case 2.2:** suppose that p divides xy for any $p \in D_{pr}(n)$. Let $m = \prod_{p \in D_{pr}(n)} p$. Since x and y are not isolated, by Corollary 2.1, m does not divide x and m does not divide y. It follows that there exist $p, q \in D_{pr}(n)$ such that p does not divide x and q does not divide y. Since p and q are prime integers, we have $gcd(x, p) = 1$ and $gcd(y, q) = 1$; namely, x is adjacent with p and y is adjacent with q. If $p \neq q$, then we get that $gcd(p, q) = 1$; namely, p is adjacent with q. In this case, $d(x, y) \leq 3$. On the other hand, if $d(x, y) = 2$, then there exists a vertex z adjacent with both x and y; namely, $gcd(z, x) = 1$ and $gcd(z, y) = 1$. Let p' be a prime divisor of z, then p' divides neither x nor y; namely, p' does not divide xy. This contradicts our assumption. Thus, if $p \neq q$, then $d(x, y) = 3$. Otherwise, if it is impossible to find such p and q, where $p \neq q$, then x and y have the same prime divisors. By our assumption, x and y are divisible by m; namely, they are isolated, which is absurd. □

Corollary 2.2. *Let* n *be a composite integer. Then, the diameter of* Γ_n *is given as follows:*

$$diam(\Gamma_n) = \begin{cases} 1 & \text{if } n = p_1 p_2, \text{ where } p_1, p_2 \in D_{pr}(n), \\ 2 & \text{if } n = p_1 p_2 p_3, \text{ where } p_1, p_2, p_3 \in D_{pr}(n), \\ 3 & \text{if } n = p_1 p_2 \cdots p_r, \text{ where } p_1, p_2, \ldots, p_r \in D_{pr}(n) \text{ and } r > 3, \\ +\infty & \text{otherwise.} \end{cases}$$

Proof. By Theorem 2.2, we have $diam(\Gamma_n) = 1$ if $n = p_1p_2$, where $p_1, p_2 \in D_{pr}(n)$, and $diam(\Gamma_n) = 2$ if $n = p_1p_2p_3$, where $p_1, p_2, p_3 \in D_{pr}(n)$. Suppose that $n = p_1p_2 \cdots p_r$, where $p_1, p_2, \ldots, p_r \in D_{pr}(n)$ and $r > 3$. Then, we take $x = p_1p_2p_3$ and $y = p_3p_4 \ldots p_r$. By Theorem 2.2, we have $d(x, y) = 3$. Thus, $diam(\Gamma_n) = 3$. In the other cases, we get always $Iso(n) \neq \emptyset$; namely, Γ_n is disconnected. Thus, $diam(\Gamma_n) = +\infty$. □

Next, we compute the girth of Γ_n.

Proposition 2.3. *Let n be a composite integer. Then, the girth of Γ_n is given as follows:*

$$girth(\Gamma_n) = \begin{cases} 4 & \text{if } n = p_1^{v_1}p_2^{v_2}, \text{ where } p_1, p_2 \in D_{pr}(n) \text{ and } \min(v_1, v_2) \geq 2, \\ 3 & \text{if } \nu(n) \geq 3, \\ +\infty & \text{otherwise.} \end{cases}$$

Proof. Let $n = p_1^{v_1}p_2^{v_2}$, where p_1 and p_2 are prime integers and $\min(v_1, v_2) \geq 2$. Let x be a non-isolated vertex of Γ_n. Then, either $x = p_1^r$, where $r \leq v_1$, or $x = p_2^r$, where $r \leq v_2$. For example, say that $x = p_1^r$, where $r \leq v_1$. Then, $V_n(x) = \{p_2^k \mid 0 < k \leq v_2\}$. Notice that, for any element $y = p_2^s$ in $V_n(x)$, we have $V_n(y) = \{p_1^k \mid 0 < k \leq v_1\}$. Since $v_1 \geq 2$, there exists $u \neq r$ such that $z = p_1^u \in V_n(y)$. As well, since $v_2 \geq 2$, there exists $w \neq s$ such that $t = p_2^w \in V_n(z) = V_n(x)$. It follows that we get a 4-cycle (x, y, z, t), which is the smallest. Thus, $girth(\Gamma_n) = 4$. Suppose now that $\nu(n) \geq 3$. then, there exist p_1, p_2 and p_3 three prime divisors of n. This provides a 3-cycle (p_1, p_2, p_3). Thus, $girth(\Gamma_n) = 3$. In the other cases, we have either $\nu(n) = 1$ or $n = p_1^v.p_2$ where p_1 and p_2 are two prime integers and $v \geq 1$. If $\nu(n) = 1$, then the graph Γ_n is empty, then $girth(\Gamma_n) = +\infty$. If $n = p_1^v.p_2$, then any vertex divisible by p_1 is adjacent only with p_2, then it is impossible to get a cycle. Thus, $girth(\Gamma_n) = +\infty$. □

The independence number $\alpha(G)$ of a graph G is defined as the size of the largest subset $I \subseteq V(G)$ such that no two vertices in I are adjacent. A subset $I \subseteq V(G)$ is independent if there is no pair of adjacent vertices in I.

Theorem 2.3. *Let n be a composite integer and $\mathfrak{p} \in D_{pr}(n)$ such that $v_{\mathfrak{p}}(n) = \max\{v_p(n) \mid p \in D_{pr}(n)\}$. Then, the independence number of $\Gamma(n)$ is given as follows:*

$$\alpha(n) = v_{\mathfrak{p}}(n) \times \prod_{\substack{p \in D_{pr}(n) \\ p \neq \mathfrak{p}}} (v_p(n) + 1) - 1.$$

Proof.

- **Step 1.** Set $m = \prod_{p \in D_{pr}(n)} p$. Let us prove that $J_q = \{x \in D(m) \mid q \text{ divides } x\}$ is a maximal independent subset of $D(m)$, where $q \in D_{pr}(n)$. First, notice that

$$J_q = \left\{ q \times \prod_{p \in D_{pr}(n) \setminus \{q\}} p^{v_p} \mid v_p \in \{0, 1\}, \forall p \in D_{pr}(n) \setminus \{q\} \right\} \setminus \{m\}.$$

  Then, $|J_q| = 2^{r-1} - 1$, where $r = \nu(n)$. Suppose that I is an independent subset of $D(m)$ such that $|I| \geq 2^{r-1}$. Let $I' = \{\frac{m}{x} \mid x \in I\}$. Then, it is easy to see that $I' \subseteq D(m)$. Moreover, for any $p \in D_{pr}(x)$, we have $v_p(\frac{m}{x}) = v_p(m) - v_p(x) = 1 - 1 = 0$, for each $x \in I$. Thus, $gcd(\frac{m}{x}, x) = 1$, for any $x \in I$; namely, x and $\frac{m}{x}$ are adjacent. Since I is independent, we get that $I \cap I' = \emptyset$. It follows that $x \in I \mapsto \frac{m}{x} \in I'$ is a one-to-one correspondence between I and I'. So that $|I'| = |I| \geq 2^{r-1}$. On the other hand, we have $I \cup I' \subseteq D(m)$. Then, $|I \cup I'| \leq |D(m)|$. Recall that $|D(m)| = 2^r - 2$, and since I and I' are disjoint, we have $|I \cup I'| = |I| + |I'| \geq 2 \times 2^{r-1} = 2^r$, which contradicts the fact that $|I \cup I'| \leq |D(m)|$. Hence, J_q is a maximal independent subset of $D(m)$.

- **Step 2.** Let $q \in D_{pr}(n)$ and $\langle q \rangle$ be a maximal independent subset in $D(n)$ containing q. Let us prove that $\langle q \rangle = \{x \in D(n) \mid x \text{ is divisible by } q\}$. If there is $x \in \langle q \rangle$ such that q does not divide x, then $gcd(q, x) = 1$ since q is prime. It follows that q and x are adjacent, which contradicts the fact that $\langle q \rangle$ is independent. Therefore, $\langle q \rangle \subseteq \{x \in D(n) \mid x \text{ is divisible by } q\}$. As well, $|\langle q \rangle| \leq |\{x \in D(n) \mid x \text{ is divisible by } q\}|$. It suffices now to prove that $\{x \in D(n) \mid x \text{ is divisible by } q\}$ is independent. Indeed, if $x, y \in \{x \in D(n) \mid x \text{ is divisible by } q\}$, then q divides both x and y; namely, x and y are not adjacent. Thus, $\{x \in D(n) \mid x \text{ is divisible by } q\}$ is independent. Since $\langle q \rangle$ is a maximal independent subset of $D(n)$ containing q, we get that $|\langle q \rangle| \geq |\{x \in D(n) \mid x \text{ is divisible by } q\}|$. Therefore, we have

$$\begin{cases} \langle q \rangle \subseteq \{x \in D(n) \mid x \text{ is divisible by } q\}, \\ |\langle q \rangle| = |\{x \in D(n) \mid x \text{ is divisible by } q\}|. \end{cases}$$

  Therefore, $\langle q \rangle = \{x \in D(n) \mid x \text{ is divisible by } q\}$.

- **Step 3.** Let A be a subset of $D(n)$. We take $A_m = A \cap D(m)$. Also, for any $x \in A_m$, we take

$$cov(x) = \left\{ x \times \prod_{p \in D_{pr}(x)} p^{r_p} \mid r_p \in \{0, 1, ..., v_p(n) - 1\}, \text{ for any } p \in D_{pr}(x) \right\}.$$

Let us prove that, for any maximal independent subset A of $D(n)$, there is an independent subset T of $D(m)$ such that $A_m = T$ and $A = \bigcup_{x \in A_m} cov(x) \cup Iso(n)$. Notice that A_m is independent since it is a subset of A, which is independent. Also, A_m is a subset of $D(m)$, and hence $A_m = T$ is an independent subset of $D(m)$. Let $a \in A$. Since $a = \prod_{p \in D_{pr}(a)} p^{v_p(a)}$, if $D_{pr}(a) = D_{pr}(n)$, then $x \in Iso(n)$. Suppose that $D_{pr}(a) \neq D_{pr}(n)$. Let us prove that $x = \prod_{p \in D_{pr}(a)} p \in A_m$. Indeed, $x \in D(m)$ since $v_p(x) \leq 1$, for any $p \in D_{pr}(n)$. Moreover, suppose that $x \notin A$. Since A is a maximal independent subset of $D(n)$, $A \cup \{x\}$ is not independent. It follows that x is adjacent with a certain $y \in A$; namely, $gcd(x, y) = 1$. But, we have $D_{pr}(x) = D_{pr}(a)$, then for any $p \in D_{pr}(a)$, we have $p \in D_{pr}(x)$; namely, $v_p(x) \geq 1$. Since $gcd(x, y) = 1$, we get that $v_p(y) = 0$. Thus, $gcd(a, y) = 1$; namely, a and y are adjacent. This contradicts the fact that $a, y \in A$ and A is independent. Therefore, $a \in cov(x)$, with $x \in A_m$. Thus, we get that $A \subseteq \bigcup_{x \in A_m} cov(x)$. On the other hand, let $x \in A_m$ and $a \in cov(x)$. Then, $a = x \times \prod_{p \in D_{pr}(x)} p^{r_p}$, where $r_p \in \{0, 1, ..., v_p(n) - 1\}$, for any $p \in D_{pr}(x)$. Let $y \in A$. Since A is independent, we get that y is not adjacent with x; namely, there is a prime $p \in D_{pr}(x)$, which divides y. Notice that $D_{pr}(x) = D_{pr}(a)$. Then, $p \in D_{pr}(a)$ and divides y. Therefore, a is not adjacent with y, for any $y \in A$. If $a \notin A$, we get that $A \cup \{a\}$ is independent, which contradicts the fact that A is maximal. Therefore, $a \in A$. Hence, $A = \bigcup_{x \in A_m} cov(x) \cup Iso(n)$.

- **Step 4.** Since J_q are maximal independent subsets of $D(m)$, we get a maximal independent subset I of $D(n)$ of the form $I = \bigcup_{x \in J_q} cov(x) \cup Iso(n)$, for some $q \in D_{pr}(n)$. As well, since $q \in J_q = I_m$, we get that I contains q; namely, $I = \langle q \rangle$. By Step 2, we have $\langle q \rangle = \{x \in D(n) \mid x \text{ is divisible by } q\}$, for any $q \in D_{pr}(n)$. It follows that

$$\langle q \rangle = \left\{ q \times \prod_{p \in D_{pr}(n)} p^{s_p} \mid 0 \leq s_q \leq v_q(n) - 1, 0 \leq s_p \leq v_p(n), \forall p \in D_{pr}(n) \setminus \{q\} \right\} \setminus \{n\}.$$

Then,

$$|\langle q \rangle| = v_q(n) \times \prod_{\substack{p \in D_{pr}(n) \\ p \neq q}} (v_p(n) + 1) - 1.$$

It follows that the independence number of Γ_n is

$$\alpha(n) = \max \left\{ v_q(n) \times \prod_{\substack{p \in D_{pr}(n) \\ p \neq q}} (v_p(n) + 1) - 1 \mid q \in D_{pr}(n) \right\}.$$

Let $\mathfrak{p} \in D_{pr}(n)$ such that $v_{\mathfrak{p}}(n) = \max\{v_p(n) \mid p \in D_{pr}(n)\}$. Then, for any $q \in D_{pr}(n)$, we have:

$$\begin{aligned} |\langle \mathfrak{p} \rangle| - |\langle q \rangle| &= \Big(v_{\mathfrak{p}}(n) \times \prod_{\substack{p \in D_{pr}(n) \\ p \neq \mathfrak{p}}} (v_p(n) + 1) - 1 \Big) - \Big(v_q(n) \times \prod_{\substack{p \in D_{pr}(n) \\ p \neq q}} (v_p(n) + 1) - 1 \Big), \\ &= \prod_{\substack{p \in D_{pr}(n) \\ p \notin \{\mathfrak{p}, q\}}} (v_p(n) + 1) \times \big(v_{\mathfrak{p}}(n)(v_q(n) + 1) - v_q(n)(v_{\mathfrak{p}}(n) + 1) \big), \\ &= \prod_{\substack{p \in D_{pr}(n) \\ p \notin \{\mathfrak{p}, q\}}} (v_p(n) + 1) \times \big(v_{\mathfrak{p}}(n) - v_q(n) \big). \end{aligned}$$

Since $v_{\mathfrak{p}}(n) = \max\{v_p(n) \mid p \in D_{pr}(n)\}$, we get that $v_{\mathfrak{p}}(n) - v_q(n) \geq 0$; namely, $|\langle \mathfrak{p} \rangle| - |\langle q \rangle| \geq 0$. Thus,

$$|\langle \mathfrak{p} \rangle| = \max\{|\langle p \rangle \mid p \in D_{pr}(n)\}.$$

Therefore,

$$\alpha(n) = v_{\mathfrak{p}}(n) \times \prod_{\substack{p \in D_{pr}(n) \\ p \neq \mathfrak{p}}} (v_p(n) + 1) - 1.$$

□

Remark 2.3. *By the proof of Theorem 2.3, the set* $\{x \in D(n) \mid x \text{ is divisible by } \mathfrak{p}\}$ *is a maximal independent subset of* $D(n)$.

Example 2.2. *Let* $n = 2^3 \times 3 \times 5 = 120$. *We have* $v_2(n) = 3$ *and* $v_3(n) = v_5(n) = 1$, *then* $\mathfrak{p} = 2$. *It follows that* $\alpha(n) = 3 \times 2 \times 2 = 12$. *Furthermore, we have* $\langle 2 \rangle = \{2, 4, 6, 8, 10, 12, 20, 24, 30, 40, 48, 60\}$ *is a maximal independent subset of* $D(120)$.

Next, we compute the chromatic number and the clique number of Γ_n, and prove that Γ_n is perfect.

Theorem 2.4. *Let* n *be a composite integer. Then, the chromatic number of* Γ_n *is the number of prime divisors of* n*; namely,*

$$\chi(\Gamma_n) = \nu(n).$$

Proof. By Proposition 2.1, $D_{pr}(n)$ forms a clique in Γ_n. Hence, we have

$$\chi(\Gamma_n) \geq \nu(n). \tag{1}$$

On the other hand, suppose that $D_{pr}(n) = \{p_1, p_2, ..., p_r\}$, where $r = \nu(n)$. Then, for any $j \in \{1, 2, ..., r\}$, we define the set M_{p_j} as follows:

$$M_{p_j}(n) = \left\{ d \in D\left(\prod_{i=j}^{r} p_i^{v_{p_i}(n)}\right) \mid p_j \text{ divides } d \right\}.$$

- **Step 1.** Let us prove that the family $(M_p(n))_{p \in D_{pr}(n)}$ is a partition of the set $D(n)$.
  - Let us prove that $D(n) = \bigcup_{p \in D_{pr}(n)} M_p(n)$. It is obvious that $\bigcup_{p \in D_{pr}(n)} M_p(n) \subseteq D(n)$. Let $d \in D(n)$ and $j = \min\{i \in \{1, 2, ..., r\} \mid p_i \text{ divides } d\}$. Then, $d \in D\left(\prod_{i=j}^{r} p_i^{v_{p_i}(n)}\right)$ and p_j divides d; namely, $d \in M_{p_j}(n)$. Therefore, we get that
  $$D(n) = \bigcup_{p \in D_{pr}(n)} M_p(n).$$
  - Let us prove that $M_{p_k}(n) \cap M_{p_j}(n) = \emptyset$, where $k < j$. Let $d \in M_{p_j}(n)$. Then, $d \in D\left(\prod_{i=j}^{r} p_i^{v_{p_i}(n)}\right)$; namely, d divides $\prod_{i=j}^{r} p_i^{v_{p_i}(n)}$. It follows that $v_{p_k}(d) \leq v_{p_k}\left(\prod_{i=j}^{r} p_i^{v_{p_i}(n)}\right) = 0$. So that $v_{p_k}(d) = 0$. Then, p_k does not divide d. Consequently, $d \notin M_{p_k}(n)$. Thus,
  $$M_{p_k}(n) \cap M_{p_j}(n) = \emptyset.$$

  Therefore, the family $(M_p(n))_{p \in D_{pr}(n)}$ is a partition of the set $D(n)$.
- **Step 2.** Let $p \in D_{pr}(n)$. Notice that if $d, d' \in M_p(n)$ such that $d \neq d'$, then d and d' are not adjacent since they are both divisible by p. As a result, each $M_p(n)$ can be assigned a unique distinct color, and the total number of colors used will be at least $\nu(n)$. Since $(M_p(n))_{p \in D_{pr}(n)}$ is a partition of the set $D(n)$, we get that all the graph has been colored using $\nu(n)$ colors, where any two vertices colored with the same color are not adjacent. Equivalently, $\chi(\Gamma_n) \leq \nu(n)$. Therefore, by (1), we have $\chi(\Gamma_n) = \nu(n)$. □

Notice that the proof of the previous result describes also the coloring strategy. Let us now prove that Γ_n is perfect.

Theorem 2.5. *Let* n *be a composite integer. Then, the graph* Γ_n *is perfect.*

Proof. Let G be a subgraph of Γ_n and $V(G)$ the set of vertices of G. If $|V(G)| = 1$, then it is obvious that $\chi(G) = \omega(G) = 1$. Suppose that $|V(G)| = 2$; namely, $V(G) = \{x, y\}$, for some $x, y \in D(n)$.

If x and y are coprime, then x and y are adjacent. It follows that $\omega(G) = \chi(G) = 2$.

If x and y are not coprime, then x and y are not adjacent. It follows that $\omega(G) = \chi(G) = 1$.

By induction, suppose that $\chi(G) = \omega(G)$, for any subgraph G of Γ_n such that $|V(G)| < |D(n)|$. Let $v \in D(n) \setminus V(G)$, and $G' = G \cup \{v\}$. Let us prove that $\chi(G') = \omega(G')$. We distinguish it into two cases:

- **Case 1:** suppose that there exists a maximal clique $\mathcal{C}$ of G such that v is adjacent with x, for any $x \in \mathcal{C}$. In this case, it is obvious that $\mathcal{C} \cup \{v\}$ forms a maximal clique of G'. Therefore, $\omega(G') = \omega(G) + 1$. On the other hand, since $\omega(G) = \chi(G)$ and v is adjacent with every other vertex of $\mathcal{C}$, the vertex v should be colored with a new color; namely, $\chi(G') = \chi(G) + 1$. Thus, $\chi(G') = \omega(G')$.
- **Case 2:** suppose that, for any maximal clique $\mathcal{C}$ of G, there exists $x \in \mathcal{C}$ such that v is not adjacent with x. In this case, $\omega(G) = \omega(G')$. On the other hand, there is a prime integer p, which divides both v and x. Then, we can associate to v the same color of $M_p(n)$. So we get that $\chi(G') = \chi(G)$. Thus, $\chi(G') = \omega(G')$.

We conclude that Γ_n is perfect. □

Now, we get also the clique number of Γ_n.

Corollary 2.3. *Let n be a composite integer. Then, $\omega(\Gamma_n) = \nu(n)$.*

Proof. By Theorem 2.5, we have $\chi(\Gamma_n) = \omega(\Gamma_n)$. Also, by Theorem 2.4, we have $\chi(\Gamma_n) = \nu(n)$. Therefore, $\omega(\Gamma_n) = \nu(n)$. □

Now, from the Perfect Graph Theorem [6], the next result follows.

Corollary 2.4. *Let n be a composite integer. Then, the complement graph of Γ_n is perfect.*

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