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Research Article

Modular irregularity strength of the corona product of graphs

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Abstract

Let $G(V,E)$ be a graph of order n. A modular irregular labeling of G is an edge k-labeling $\phi : E(G) \to \{1,2,\dots,k\}$ provided that the weight function $\sigma : V(G) \to \mathbb{Z}_n$ defined by $\sigma(u) = wt_\phi(u) = \sum_{uv \in E(u)} \phi(uv) \pmod n$ is bijective, where $E(u)$ denotes the set of all those edges in $E(G)$ that are incident with the vertex u and $\mathbb{Z}_n$ is the group of integers modulo n. This weight function is called a modular weight of the vertex u. The minimum number k such that the graph G has a modular irregular labeling with the largest label k is called the modular irregularity strength of G. In this paper, we determine the modular irregularity strength of the corona product of a graph G with the edgeless graph of order p (that is, the graph consisting of p isolated vertices) and with the path graph P_3 of order 3, where G is a regular graph containing a 1-factor.

Keywords: modular irregularity strength; modular irregular labeling; regular graph; corona product of graphs.

2020 Mathematics Subject Classification: 05C78, 05C15.

1. Introduction

In 1988, Chartrand et al. [7] introduced the concept of an irregular labeling. An irregular labeling of a simple graph G is an edge labeling $\phi : E(G) \to \{1,2,\dots,k\}$, where k is a positive integer, such that every two distinct vertices have distinct weights. The weight of a vertex $u \in V(G)$ is defined by $wt_\phi(u) = \sum_{uv \in E(u)} \phi(uv)$, where $E(u)$ denotes the set of all the edges in $E(G)$ incident with the vertex u. The irregularity strength of G, denoted by $\mathrm{s}(G)$, is the minimum number k for which the graph G admits an irregular labeling with the label at most k. If there exists no such labeling for the graph G, then $\mathrm{s}(G) = \infty$. It is clear that $\mathrm{s}(G) < \infty$ if and only if G contains no isolated edges and has at most one isolated vertex. Chartrand et al. [7] also give a lower bound for the irregularity strength of a graph:

Theorem 1.1. [7] *If G is a connected graph of order at least 3 containing n_i vertices of degree i, then*

$$\mathrm{s}(G) \geq \max_{1 \leq i \leq \Delta(G)} \left\{ \frac{n_i - 1}{i} + 1 \right\}.$$

Other upper bounds for the irregularity strength have been derived in [2,9–11]. There are many variations of irregular labeling in the literature; for instance, edge irregular labeling, total vertex irregular labeling, total edge irregular labeling [1,4], etc. Another variation of irregular labeling is the modular irregular labeling, which was introduced by Bača et al. [6]. A modular irregular labeling of a graph G of order n is an edge k-labeling $\phi : E(G) \to \{1,2,\dots,k\}$ provided that the weight function $\sigma : V(G) \to \mathbb{Z}_n$ defined by $\sigma(u) = wt_\phi(u) = \sum_{uv \in E(u)} \phi(uv) \pmod n$ is bijective, where $E(u)$ denotes the set of all those edges in $E(G)$ that are incident with the vertex u and $\mathbb{Z}_n$ is the group of integers modulo n. This weight function is called the modular weight of the vertex u. The minimum number k such that the graph G admits a modular irregular labeling with the largest label k is called the modular irregularity strength of G, denoted by $\mathrm{ms}(G)$. If G admits no modular irregular labeling, it is defined as $\mathrm{ms}(G) = \infty$.

Bača et al. [6] proved a sufficient condition for the modular irregularity strength of a graph to be infinite. They also proved a relation between the irregularity strength of a graph G and its modular irregularity strength:

Theorem 1.2. [6] *If G is a graph of order n such that $n \equiv 2 \pmod 4$, then G has no modular irregular k-labeling, i.e.,* $\mathrm{ms}(G) = \infty$.

Theorem 1.3. [6] *Let G be a graph with no component of order 1 or 2. Then $\mathrm{s}(G) \leq \mathrm{ms}(G)$.*

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There are some classes of graphs for which the modular irregularity strength has been determined. In [6], Bača et al. determined the modular irregularity strength of paths, stars, triangular graphs, cycles, and gear graphs. Later in 2021, Bača et al. determined the modular irregularity strength of fan graphs and wheels [3,5]. In the same year, Sugeng et al. [12] determined the modular irregularity strength of double-stars and friendship graphs. In 2022, Dewi [8] determined the modular irregularity strength of the corona product of the cycle graph of order n and p isolates, denoted by $C_n \odot \overline{K_p}$. Recently, in 2023, Sugeng et al. [13] determined the modular irregularity strength of some flower graphs.

The corona product of two graphs G and H, denoted by $G \odot H$, is defined as the graph obtained by taking one copy of G and $|V(G)|$ copies of H and joining the ith vertex of G to every vertex in the ith copy of H. Let $\overline{G}$ denote the complement of a graph G. In this paper, we determined the modular irregularity strength of $G \odot \overline{K_p}$ and $G \odot P_3$, where G is a regular graph containing a 1-factor.

2. Corona product of a graph and isolated vertices

In this section, we discuss the modular irregularity strength of the corona product of graphs G and $\overline{K_p}$. First, we deal with the corona product of a regular graph G and $\overline{K_p}$ when p is odd.

Theorem 2.1. *Let G be a regular graph of order n containing a 1-factor. If p is odd then*

$$\mathrm{ms}(G \odot \overline{K_p}) = np.$$

Proof. Let G be an r-regular graph of order n containing a 1-factor $M(G)$. Thus n is even. Let v_i, $i = 1, 2, \dots, n$ be vertices of a graph G and let $x_{i,j}$, $i = 1, 2, \dots, n$, $j = 1, 2, \dots, p$ be the pendant vertices in $G \odot \overline{K_p}$ that are adjacent to v_i.

For p odd, we define an edge labeling φ of $G \odot \overline{K_p}$ in the following way.

$$\varphi(v_i x_{i,j}) = \begin{cases} jn - n + i, & \text{if } 1 \le i \le n \text{ and } j \text{ is odd, } 1 \le j \le p, \\ jn + 1 - i, & \text{if } 1 \le i \le n \text{ and } j \text{ is even, } 2 \le j \le p-1, \end{cases} \tag{1}$$

$$\varphi(e) = \begin{cases} \left\lfloor \frac{np - \frac{p-1}{2}}{r} \right\rfloor, & \text{if } e \in E(G) \setminus M(G), \\ np - \frac{p-1}{2} - (r-1)\left\lfloor \frac{np - \frac{p-1}{2}}{r} \right\rfloor, & \text{if } e \in M(G). \end{cases} \tag{2}$$

As

$$\max\{\varphi(v_i x_{i,j}) : 1 \le i \le n, 1 \le j \le p\} = np, \tag{3}$$

$$\max\{\varphi(e) : e \in E(G)\} \le \left\lceil \frac{np - \frac{p-1}{2}}{r} \right\rceil < np, \tag{4}$$

we get that φ is an np-labeling.

Now, we check the vertex weights. First, we evaluate the weights of the pendant vertices.

$$wt_\varphi(x_{i,j}) = \varphi(v_i x_{i,j}) = \begin{cases} jn - n + i, & \text{if } 1 \le i \le n \text{ and } j \text{ is odd, } 1 \le j \le p, \\ jn + 1 - i, & \text{if } 1 \le i \le n \text{ and } j \text{ is even, } 2 \le j \le p-1. \end{cases} \tag{5}$$

Hence, the set of the corresponding modular weights consists of integers from 1 up to np.

Let us denote by $E_G(v_i)$ the set of all edges of the graph G incident with vertex v_i, $i = 1, 2, \dots, n$. Then the weight of the vertex v_i is

$$wt_\varphi(v_i) = \sum_{e \in E_G(v_i)} \varphi(e) + \sum_{j=1}^{p} \varphi(v_i x_{i,j}). \tag{6}$$

If e_i is the edge from $E_G(v_i)$ belonging to $M(G)$, then

$$\sum_{e \in E_G(v_i)} \varphi(e) = \sum_{\substack{e \in E_G(v_i) \\ e \neq e_i}} \varphi(e) + \varphi(e_i) = (r-1)\left\lfloor \frac{np - \frac{p-1}{2}}{r} \right\rfloor + \left(np - \frac{p-1}{2} - (r-1)\left\lfloor \frac{np - \frac{p-1}{2}}{r} \right\rfloor \right) = np - \frac{p-1}{2}.$$

We also have

$$\sum_{j=1}^{p}\varphi(v_ix_{i,j})=\sum_{\substack{j=1\\ j\text{ odd}}}^{p}\varphi(v_ix_{i,j})+\sum_{\substack{j=2\\ j\text{ even}}}^{p-1}\varphi(v_ix_{i,j})=\sum_{\substack{j=1\\ j\text{ odd}}}^{p}(jn-n+i)+\sum_{\substack{j=2\\ j\text{ even}}}^{p-1}(jn+1-i)$$

$$=n\sum_{j=1}^{p}j+(i-n)\tfrac{p+1}{2}+(1-i)\tfrac{p-1}{2}=\tfrac{n(p-1)(p+1)}{2}+\tfrac{p-1}{2}+i.$$

Putting in (6), we get

$$wt_\varphi(v_i)=\left(np-\tfrac{p-1}{2}\right)+\left(\tfrac{n(p-1)(p+1)}{2}+\tfrac{p-1}{2}+i\right)=np+\tfrac{n(p-1)(p+1)}{2}+i.$$

Moreover, since p is odd, we have

$$wt_\varphi(v_i)\equiv np+i \pmod{n(p+1)}.$$

Hence, the set of modular weights of vertices v_i, $i=1,2,\dots,n$, is

$$\{wt_\varphi(v_i): i=1,2,\dots,n\}=\{0,np+1,np+2,\dots,n(p+1)-1\}. \tag{7}$$

Combining (5) and (7), we get that the set of all modular weights is

$$\{wt_\varphi(v): v\in V(G\odot\overline{K_p})\}=\{0,1,\dots,n(p+1)-1\}.$$

Thus $\mathrm{ms}(G\odot\overline{K_p})\le np$.

On the other hand, the modular irregularity strength of $G\odot\overline{K_p}$ is at least np because the graph $G\odot\overline{K_p}$ has np pendants and each pendant must have different weight, i.e., $\mathrm{ms}(G\odot\overline{K_p})\ge np$. Thus, we conclude that $\mathrm{ms}(G\odot\overline{K_p})=np$. □

By Theorem 2.1, we obtain the next results.

Corollary 2.1. *Let G be a bridgeless cubic graph of order n. If p is odd then*

$$\mathrm{ms}(G\odot\overline{K_p})=np.$$

Proof. According to Petersen's Theorem, every cubic graph with no bridges has a perfect matching. The proof follows from Theorem 2.1. □

A *sun graph* $C_n\odot K_1$ is defined as the graph obtained from a cycle C_n by adding a pendant edge to every vertex in the cycle. When n is odd then the order of the sun graph $C_n\odot K_1$ is congruent to 2 modulo 4 and thus by Theorem 1.2 we have $\mathrm{ms}(C_n\odot K_1)=\infty$. For n even we get the following for the disjoint union of m sun graphs.

Corollary 2.2. *Let $\cup_{j=1}^{m}(C_{n_j}\odot K_1)$ be a disjoint union of m sun graphs $C_{n_j}\odot K_1$, each having even order n_j. Then for every positive integer m the following holds:*

$$\mathrm{ms}\left(\bigcup_{j=1}^{m}(C_{n_j}\odot K_1)\right)=\sum_{j=1}^{m}n_j.$$

Proof. The proof is an immediate consequence of Theorem 2.1 because when n_j is even for every j, the cycle C_{n_j} contains a perfect matching. □

It is possible to prove a more general result.

Corollary 2.3. *Let G be a bipartite k-regular graph of order n. If p is odd then*

$$\mathrm{ms}(G\odot\overline{K_p})=np.$$

Proof. Let G be a bipartite k-regular graph of order n. Note that G does not need to be connected. As G is bipartite, it contains only even cycles and thus it has a 1-factor. The result follows from Theorem 2.1. □

In [8], the modular irregularity strength for $C_n\odot\overline{K_p}$ is already calculated.

Theorem 2.2. [8] *For $n\ge 3$ and $p\ge 1$, let $C_n\odot\overline{K_p}$ be a corona product of C_n and $\overline{K_p}$ of order $n(p+1)$ where $n(p+1)\not\equiv 2 \pmod 4$. Then $\mathrm{ms}\left(C_n\odot\overline{K_p}\right)=np$.*

Observe that in Theorem 2.2, there is no requirement for p to be odd or even. The only condition is for the order of the graph, that is $n(p+1) \not\equiv 2 \pmod 4$ because in this case by Theorem 1.2, the graph has no modular irregular labeling. Theorem 2.1 is a generalization of this result for the case when p is odd.

Let $n, m,$ and $a_1, a_2, \dots, a_m$ be positive integers, $1 \le a_1 < a_2 < \cdots < a_m \le \lfloor \frac{n}{2} \rfloor$. An undirected graph with the set of vertices $V = \{v_i : 1 \le i \le n\}$ and the set of edges $E = \{v_i v_{i+a_k} : 1 \le i \le n, 1 \le j \le m\}$, the indices being taken modulo n, is called a *circulant graph* and it is denoted by $C_n(a_1, a_2, \dots, a_m)$.

Theorem 2.3. *Let $C_n(1, a_2, \dots, a_m)$ be a circulant graph of even order n, $n \ge 4$, and let p be a positive integer. If $n(p+1) \not\equiv 2 \pmod 4$, then*

$$\mathrm{ms}\left(C_n(1, a_2, \dots, a_m) \odot \overline{K_p}\right) = np.$$

Proof. Let

$$\begin{aligned} V(C_n(1, a_2, \dots, a_m) \odot \overline{K_p}) &= \{v_i, x_{i,j} : 1 \le i \le n, 1 \le j \le p\} \\ E(C_n(1, a_2, \dots, a_m) \odot \overline{K_p}) &= \{v_i v_{i+a_k (\text{mod } n)}, v_i x_{i,j} : 1 \le i \le n, 1 \le j \le p, 1 \le k \le m\} \end{aligned} \tag{8}$$

be the vertex set and edge set of $C_n(1, a_2, \dots, a_m) \odot \overline{K_p}$, respectively. Since n is even, the circulant graph $C_n(1, a_2, \dots, a_m)$ is a regular graph containing a 1-factor. When p is odd, the result follows from Theorem 2.1.

Now, consider the case when p is even. Note that in this case, n must be divisible by 4. Define an edge labeling ψ in the following way

$$\psi(v_i x_{i,j}) = \begin{cases} jn - n + i, & \text{if } 1 \le i \le n \text{ and } j \text{ is odd, } 1 \le j \le p-1, \\ jn + 1 - i, & \text{if } 1 \le i \le n \text{ and } j \text{ is even, } 2 \le j \le p, \end{cases}$$

$$\psi(v_i v_{i+a_k}) = 1, \quad \text{if } 1 \le i \le n,\ 2 \le k \le m,$$

$$\psi(v_i v_{i+1}) = \begin{cases} \frac{(3n-1)p}{4} - m + 1 + 2\left\lfloor \frac{i}{2} \right\rfloor, & \text{if } 1 \le i \le \frac{n}{2},\ p \equiv 0 \pmod 4 \text{ and } a_m < \frac{n}{2}, \\ \frac{(3n-1)p+2}{4} - m + i, & \text{if } 1 \le i \le \frac{n}{2},\ p \equiv 2 \pmod 4 \text{ and } a_m < \frac{n}{2}, \\ \frac{(3n-1)p}{4} - m + 1 + i, & \text{if } 1 \le i \le \frac{n}{2},\ p \equiv 0 \pmod 4 \text{ and } a_m = \frac{n}{2}, \\ \frac{(3n-1)p-2}{4} - m + 2 + 2\left\lfloor \frac{i}{2} \right\rfloor, & \text{if } 1 \le i \le \frac{n}{2},\ p \equiv 2 \pmod 4 \text{ and } a_m = \frac{n}{2}, \\ \frac{(3n-1)p}{4} + n - m + 2 - i, & \text{if } \frac{n}{2} < i \le n,\ p \equiv 0 \pmod 4 \text{ and } a_m < \frac{n}{2}, \\ \frac{(3n-1)p+2}{4} - m + 1 + 2\left\lfloor \frac{n+1-i}{2} \right\rfloor, & \text{if } \frac{n}{2} < i \le n,\ p \equiv 2 \pmod 4 \text{ and } a_m < \frac{n}{2}, \\ \frac{(3n-1)p}{4} - m + 2 + 2\left\lfloor \frac{n+1-i}{2} \right\rfloor, & \text{if } \frac{n}{2} < i \le n,\ p \equiv 0 \pmod 4 \text{ and } a_m = \frac{n}{2}, \\ \frac{(3n-1)p+2}{4} + n - m + 2 - i, & \text{if } \frac{n}{2} < i \le n,\ p \equiv 2 \pmod 4 \text{ and } a_m = \frac{n}{2}. \end{cases}$$

It is easy to verify that ψ is an np-labeling. Furthermore, we get that the weights of vertices $x_{i,j}$, $1 \le i \le n, 1 \le j \le p$ consist of consecutive integers from 1 to np, and the weights of vertices v_i, $1 \le i \le n$, form the set $\{np+1, np+2, \dots, n(p+1)\}$. Hence, we get that the set of all modular weights is

$$\{wt_\psi(v) : v \in C_n(1, a_2, \dots, a_m) \odot \overline{K_p}\} = \{0, 1, \dots, n(p+1) - 1\}.$$

Thus, $\mathrm{ms}(C_n(1, a_2, \dots, a_m) \odot \overline{K_p}) \le np$.

Since $C_n(1, a_2, \dots, a_m) \odot \overline{K_p}$ has np pendants and each pendant must have different weight, we get

$$\mathrm{ms}(C_n(1, a_2, \dots, a_m) \odot \overline{K_p}) = np,$$

as required. □

3. Corona product of a graph and a path on three vertices

In this section, we deal with the modular irregularity strength of the corona product of graphs G and P_3.

Theorem 3.1. *Let G be a regular graph of order n containing a 1-factor. Then*

$$\mathrm{ms}(G \odot P_3) = n + 1. \tag{9}$$

Proof. Let G be an r-regular graph of order n containing a 1-factor $M(G)$. Let $v_i, i = 1, 2, \ldots, n$ be vertices of a graph G and let $x_{i,j}, i = 1, 2, \ldots, n, j = 1, 2, 3$ be vertices of P_3 that are adjacent to v_i.

Define an edge labeling ϕ of $G \odot P_3$ by

$$\phi(x_{i,j}x_{i,j+1}) = \begin{cases} i, & \text{if } j = 1 \text{ and } 1 \leq i \leq n, \\ n, & \text{if } j = 1 \text{ and } 1 \leq i \leq n, \end{cases}$$

$$\phi(v_i x_{i,j}) = \begin{cases} 1, & \text{if } j = 1 \text{ and } 1 \leq i \leq n, \\ n+1, & \text{if } j = 2 \text{ and } 1 \leq i \leq n, \\ n+2-i, & \text{if } j = 3 \text{ and } 1 \leq i \leq n, \end{cases}$$

$$\phi(e) = \begin{cases} \left\lfloor \frac{2n-2}{r} \right\rfloor, & \text{if } e \in E(G) \setminus M(G), \\ 2n-2-(r-1)\left\lfloor \frac{2n-2}{r} \right\rfloor, & \text{if } e \in M(G). \end{cases}$$

As

$$\max\{\phi(x_{i,j}x_{i,j+1}) : 1 \leq i \leq n, 1 \leq j \leq 3\} = n,$$

$$\max\{\phi(v_i x_{i,j}) : 1 \leq i \leq n, 1 \leq j \leq 3\} = n+1,$$

$$\max\{\phi(e) : e \in E(G)\} \leq \left\lceil \frac{2n-2}{r} \right\rceil < n+1,$$

we get that ϕ is an $(n+1)$-labeling.

Now, we check the vertex weights. First, we evaluate the weights of the vertices of P_3.

$$wt_\phi(x_{i,1}) = \phi(x_{i,1}x_{i,2}) + \phi(v_i x_{i,1}) = i+1,$$
$$wt_\phi(x_{i,2}) = \phi(x_{i,1}x_{i,2}) + \phi(x_{i,2}x_{i,3}) + \phi(v_i x_{i,2}) = 2n+1+i,$$
$$wt_\phi(x_{i,3}) = \phi(x_{i,2}x_{i,3}) + \phi(v_i x_{i,3}) = 2n+2-i.$$

Hence, the set of the corresponding modular weights consists of integers from 2 up to $3n+1$.

Let us denote by $E_G(v_i)$ the set of all edges of the graph G incident with vertex v_i, $i = 1, 2, \ldots, n$. Then the weight of the vertex v_i is

$$wt_\phi(v_i) = \sum_{e \in E_G(v_i)} \phi(e) + \sum_{j=1}^{3} \phi(v_i x_{i,j}). \tag{10}$$

If e_i is the edge from $E_G(v_i)$ belonging to $M(G)$ then

$$\sum_{e \in E_G(v_i)} \phi(e) = \sum_{\substack{e \in E_G(v_i) \\ e \neq e_i}} \phi(e) + \phi(e_i) = (r-1)\left\lfloor \frac{2n-2}{r} \right\rfloor + \left(2n-2-(r-1)\left\lfloor \frac{2n-2}{r} \right\rfloor\right) = 2n-2.$$

We also have

$$\sum_{j=1}^{3} \varphi(v_i x_{i,j}) = \phi(v_i x_{i,1}) + \phi(v_i x_{i,2}) + \phi(v_i x_{i,3}) = 1 + (n+1) + (n+2-i) = 2n+4-i.$$

Thus, putting in (10), we get

$$wt_\varphi(v_i) = 2n-2+2n+4-i = 4n+2-i.$$

Hence, the set of modular weights of vertices v_i, $i = 1, 2, \ldots, n$ is

$$\{wt_\phi(v_i) : i = 1, 2, \ldots, n\} = \{0, 1, 3n+2, 3n+3, \ldots, 4n-1\}.$$

Consequently, we get that the set of all modular weights is

$$\{wt_\phi(v) : v \in V(G \odot P_3)\} = \{0, 1, 2, \ldots, 4n-1\}.$$

This implies that $\mathrm{ms}(G \odot P_3) \leq n+1$.

By using Theorems 1.1 and 1.3, we obtain

$$\mathrm{ms}(G \odot P_3) \geq \mathrm{s}(G \odot P_3) \geq \max\left\{\frac{2n-1}{2}+1, \frac{n-1}{3}+1, \frac{n-1}{r+3}+1\right\} = \frac{2n-1}{2}+1 = n+\frac{1}{2}. \tag{11}$$

Since $\mathrm{s}(G)$ must be an integer, we obtain

$$\mathrm{s}(G) \geq n+1. \tag{12}$$

Thus, $\mathrm{ms}(G \odot P_3) \geq n+1$. Finally, we conclude that $\mathrm{ms}(G \odot P_3) = n+1$. □

4. Conclusion

In this paper, we evaluated the exact values of the modular irregularity strength of $G \odot \overline{K_p}$ for p odd and of $G \odot P_3$, in the case when G is a regular graph containing a 1-factor. We proved that $\mathrm{ms}(G \odot \overline{K_p}) = np$ and $\mathrm{ms}(G \odot P_3) = n+1$. According to previous research, it is possible to get a similar result for a regular graph G containing a 1-factor also in the case when p is even; namely, when G is a cycle or a circulant graph. Thus, we conclude our paper with the following open problems:

Problem 4.1. *Find regular graphs G of order n for which $\mathrm{ms}(G \odot \overline{K_p}) = np$ when p is even.*

Problem 4.2. *For a regular graph G, determine the modular irregularity strength of $G \odot P_m$ when $m = 2$ or $m > 3$.*

Problem 4.3. *For any two graphs G and H, determine the modular irregularity strength of $G \odot H$.*

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