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Research Article

Set partitions with colored singleton blocks

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Abstract

In this paper, we enumerate classes of partitions of $[n] = \{1, \ldots, n\}$ in which the singleton blocks are colored using a variable or fixed number of colors. We consider, more generally, the distribution of the statistic recording the number of colored singletons on r-partitions of $[r+n]$ in which only singletons from $[r+1, r+n]$ may be colored. Among our results, it is shown by algebraic and bijective arguments that the number of partitions of $[n]$ in which a singleton block $\{x\}$ can come in one of x colors for each x is given by the n-th row sum of Lah numbers, yielding a new combinatorial interpretation for this sequence. Also, we show that the partitions of $[n]$ in which each singleton is assigned one of $s+1$ colors where s is fixed are equinumerous with the set of s-partitions of $[s+n]$. Generalizations in terms of r-partitions of both of these results and others are demonstrated.

Keywords: finite set partition; Lah distribution; singletons statistic; exponential generating function.

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1. Introduction

By a *partition* of the set $[n] = \{1, \ldots, n\}$, we mean a collection of nonempty disjoint subsets, called *blocks*, whose union is $[n]$. Let $\mathcal{P}_n$ denote the set of all partitions of $[n]$. In this paper, we will deal with the enumeration of certain classes of $\mathcal{P}_n$ wherein the singleton blocks are colored using either a fixed or a variable number of colors. There has been recent interest in enumerative combinatorics concerning statistics related to singleton blocks and counting classes of partitions of $[n]$ (and other analogous structures) satisfying certain restrictions with regard to their singletons (see, e.g., [3,4,13,14,20,21] and references contained therein). For example, in [8,10,12], various classes of partitions are studied which contain no singletons. Perhaps some of the interest in the singletons statistic stems from the fact that its distribution on $\mathcal{P}_n$ is the same as that of the parameter tracking the number of *circular successions* (i.e., occurrences of i and $i+1$ in the same block of a partition, where n and 1 are regarded as consecutive, see, e.g., [7]). A combinatorial proof of this fact which makes use of an algorithmic bijection that switches singletons for block adjacencies and vice versa is given in [3]. See [6] for other related statistics on set partitions.

By an *r-partition* of the set $[r+n]$, we mean one in which the elements of $[r]$ belong to distinct blocks. Let $\mathcal{P}_n^{(r)}$ denote the set of all r-partitions of $[r+n]$. If the order matters in which the elements within each block in a member of $\mathcal{P}_n^{(r)}$ are written (with the order of the blocks themselves being immaterial), one obtains what is known as an *r-Lah distribution*. Let $\mathcal{L}_n^{(r)}$ denote the set of all r-Lah distributions of $[r+n]$. By a *special* block within a member of $\mathcal{P}_n^{(r)}$ or $\mathcal{L}_n^{(r)}$, we mean one which contains an element of $[r]$, with all other blocks being *non-special*. The same terminology will be applied at times to distinguish the elements themselves of $[r]$ from those in $I = [r+1, r+n]$.

Let $\mathcal{L}_{n,k}^{(r)}$ for $0 \le k \le n$ denote the subset of $\mathcal{L}_n^{(r)}$ whose members contain exactly k non-special blocks (and hence $r+k$ blocks altogether). Then the cardinality of $\mathcal{L}_{n,k}^{(r)}$ is given by the r-Lah number (see, e.g., [15,17]), which we will denote here by $L_{n,k}^{(r)}$ and is given explicitly by $\frac{n!}{k!}\binom{n+2r-1}{k+2r-1}$ for all non-negative n, k and r. Let $L_n^{(r)} = \sum_{k=0}^{n} L_{n,k}^{(r)}$ (see [16]), which gives the cardinality of all members of $\mathcal{L}_n^{(r)} = \cup_{k=0}^{n} \mathcal{L}_{n,k}^{(r)}$. Note that when $r=0$, the $L_{n,k}^{(r)}$ reduce to the classical Lah numbers $L_{n,k}$ (see, e.g., [5] and entry A008297 in the OEIS [19]). The $L_n^{(r)}$ reduce when $r=0$ to the n-th row sum of Lah numbers given by

$$L_n = \sum_{k=0}^{n} \frac{n!}{k!}\binom{n-1}{k-1}$$

for $n \ge 0$, see A000262 in [19].

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Recall that the partial r-Bell polynomials (see, e.g., [11, 18]), which are denoted by $B_{n,k}^{(r)}(x_i;y_i)$ where x_i and y_i for $i\geq 1$ are series of variables, are given explicitly by

$$B_{n,k}^{(r)}(x_i;y_i)=\sum_{\Lambda(n,k,r)}\left[\frac{n!}{k_1!k_2!\cdots}\left(\frac{x_1}{1!}\right)^{k_1}\left(\frac{x_2}{2!}\right)^{k_2}\cdots\right]\left[\frac{r!}{r_0!r_1!\cdots}\left(\frac{y_1}{0!}\right)^{r_0}\left(\frac{y_2}{1!}\right)^{r_1}\cdots\right], \tag{1}$$

where $\Lambda(n,k,r)$ is the set of all non-negative integer sequences $(k_i)_{i\geq1}$ and $(r_i)_{i\geq0}$ such that $\sum_{i\geq1}k_i=k$, $\sum_{i\geq0}r_i=r$ and $\sum_{i\geq1}i(k_i+r_i)=n$. Note that $B_{n,k}^{(0)}(x_i;y_i)=B_{n,k}(x_i)$ corresponds to the classical partial Bell polynomial [1]. The $B_{n,k}^{(r)}(x_i;y_i)$ have exponential generating function (egf) given by

$$\sum_{n\geq k}B_{n,k}^{(r)}(x_i;y_i)\frac{t^n}{n!}=\frac{1}{k!}\left(\sum_{j\geq1}x_j\frac{t^j}{j!}\right)^k\left(\sum_{j\geq0}y_{j+1}\frac{t^j}{j!}\right)^r, \tag{2}$$

see [11, Corollary 4], which reduces to the well-known egf formula for $B_{n,k}(x_i)$ when $r=0$. In what follows, we will make use of several particular cases of (2) in establishing our results. For example, the sequence of r-Lah numbers $L_{n,k}^{(r)}$ coincides with the special case of $B_{n,k}^{(r)}(x_i;y_i)$ where $x_i=y_i=i!$ for all i. Hence, one has the egf formulas

$$\sum_{n\geq k}L_{n,k}^{(r)}\frac{x^n}{n!}=\frac{x^k}{k!(1-x)^{k+2r}}\quad\text{and}\quad\sum_{n\geq0}L_{n}^{(r)}\frac{x^n}{n!}=\frac{\exp\left(\frac{x}{1-x}\right)}{(1-x)^{2r}},$$

where the latter follows from the former by summing over all $k\geq0$.

In the next section, we introduce colored r-partitions of $[r+n]$ where non-special singletons $\{x\}$ come in one of x colors. We show that the number of such partitions is given by $L_n^{(r)}$, supplying both algebraic and combinatorial proofs. Thus, one gets a new combinatorial interpretation of this sequence and connection between r-partitions and r-Lah distributions. In the case when $r=0$, one has a new interpretation of the sequence A000262 for the row sum of classical Lah numbers. We compute, more generally, the distribution of the statistic tracking the number of non-special singletons (marked by y) and show that it corresponds to a natural statistic on $\mathcal{L}_n^{(r)}$. This equivalence is instrumental in explaining bijectively the $y=-1$ case of the distribution polynomial representing the sign balance of the singletons statistic.

A similar treatment is provided in the third section where non-special singletons in members of $\mathcal{P}_n^{(r)}$ are assigned one of a fixed number of colors. Among our results, we show that the r-partitions of $[r+n]$ in which non-special singletons come in one of $s+1$ colors are equinumerous with the set of $(r+s)$-partitions of $[r+s+n]$.

2. Singletons with a variable number of colors

In this section, we consider colorings of set partitions in which a singleton $\{x\}$ receives one of x possible colors. Before enumerating such partitions, recall that the (signless) r-Stirling number of the first kind (see, e.g., [2, 15]) is defined by the recurrence $c_{n+1,k}^{(r)}=c_{n,k-1}^{(r)}+(r+n)c_{n,k}^{(r)}$, $n\geq0$ and $1\leq k\leq n+1$, together with the initial values $c_{n,0}^{(r)}=r(r+1)\cdots(r+n-1)$ and $c_{0,k}^{(r)}=\delta_{0,k}$. Taking $x_i=y_i=(i-1)!$ in (2) gives the egf formula

$$\sum_{n\geq0}c_{n,k}^{(r)}\frac{x^n}{n!}=\frac{(-\ln(1-x))^k}{k!(1-x)^r}. \tag{3}$$

Given n numbers $i_1,\ldots,i_n$ and $1\leq j\leq n$, define the j-th symmetric sum $\mathcal{S}_j(i_1,\ldots,i_n)$ by

$$\mathcal{S}_j(i_1,\ldots,i_n)=\sum_{1\leq r_1<\cdots<r_j\leq n}i_{r_1}\cdots i_{r_j},$$

with $\mathcal{S}_0(i_1,\ldots,i_n)=1$. We have

$$c_{n,j}^{(r+1)}=\mathcal{S}_{n-j}(r+1,\ldots,r+n),\qquad 0\leq j\leq n. \tag{4}$$

To realize (4), we proceed inductively on n, noting that the equality holds if $j=0$ or $j=n$. If $n\geq2$ and $1\leq j\leq n-1$, then

$$c_{n,j}^{(r+1)}=c_{n-1,j-1}^{(r+1)}+(r+n)c_{n-1,j}^{(r+1)}=\mathcal{S}_{n-j}(r+1,\ldots,r+n-1)+(r+n)\mathcal{S}_{n-j-1}(r+1,\ldots,r+n-1)=\mathcal{S}_{n-j}(r+1,\ldots,r+n),$$

which completes the induction and establishes (4).

Let $v_{n,k}^{(r)}$ denote the number of r-partitions of $[r+n]$ into $r+k$ blocks in which no singleton of the form $\{z\}$, where $z\in I$ occurs. Define $v_n^{(r)}=\sum_{k=0}^n v_{n,k}^{(r)}$ for $n\geq0$. Taking $x_i=1$ if $i\geq2$, with $x_1=0$, and $y_i=1$ for all $i\geq1$ in (2), and then summing over all $k\geq0$, implies that the egf for $v_n^{(r)}$ is given by

$$\sum_{n\geq0}v_n^{(r)}\frac{x^n}{n!}=\exp\left(e^x+(r-1)x-1\right). \tag{5}$$

Let $\mathcal{K}_n^{(r)}$ denote the set of r-partitions of $[r+n]$ in which a singleton $\{x\}$ where $x \in I$ comes in one of x colors. Given $\pi \in \mathcal{K}_n^{(r)}$, let $\mu(\pi)$ denote the number of singletons in I. Define the distribution $K_n^{(r)}(y) = \sum_{\pi \in \mathcal{K}_n^{(r)}} y^{\mu(\pi)}$ for $n \geq 1$, with $K_0^{(r)} := 1$, for all $r \geq 0$.

Lemma 2.1. *If $n \geq 0$, then*

$$K_n^{(r)}(y) = \sum_{j=0}^{n} v_j^{(r)} c_{n,j}^{(r+1)} y^{n-j}. \tag{6}$$

Proof. Suppose that a member of $\mathcal{K}_n^{(r)}$ contains exactly j singleton blocks in I. Then there are $\mathcal{S}_j\{r+1,\ldots,r+n\}y^j$ possibilities for the (weighted) choice of these singletons, along with their colors. The remaining $n-j$ members of I are then arranged together with the members of $[r]$ according to an r-partition in $v_{n-j}^{(r)}$ ways. Considering all possible $0 \leq j \leq n$ gives

$$K_n^{(r)}(y) = \sum_{j=0}^{n} v_{n-j}^{(r)} \mathcal{S}_j\{r+1,\ldots,r+n\}y^j = \sum_{j=0}^{n} v_{n-j}^{(r)} c_{n,n-j}^{(r+1)} y^j,$$

by (4), and formula (6) now follows by replacing j with $n-j$. □

Define the egf for the sequence $K_n^{(r)}(y)$ for $n \geq 0$ and a fixed r by

$$G^{(r)}(x,y) = \sum_{n \geq 0} K_n^{(r)}(y) \frac{x^n}{n!}.$$

Theorem 2.1. *We have*

$$G^{(r)}(x,y) = \frac{\exp\left((1-xy)^{-1/y} - 1\right)}{(1-xy)^{(r-1)/y+r+1}}. \tag{7}$$

Proof. By (6), (3) and (5), we have

$$\begin{aligned}
\sum_{n \geq 0} K_n^{(r)}(y) \frac{x^n}{n!} &= \sum_{n \geq 0} \frac{x^n}{n!} \sum_{j=0}^{n} v_j^{(r)} c_{n,j}^{(r+1)} y^{n-j} = \sum_{j \geq 0} v_j^{(r)} y^{-j} \sum_{n \geq j} c_{n,j}^{(r+1)} \frac{(xy)^n}{n!} \\
&= \sum_{j \geq 0} v_j^{(r)} y^{-j} \frac{(-\ln(1-xy))^j}{j!(1-xy)^{r+1}} = \frac{1}{(1-xy)^{r+1}} \sum_{j \geq 0} \frac{v_j^{(r)} \left(-\frac{\ln(1-xy)}{y}\right)^j}{j!} \\
&= \frac{1}{(1-xy)^{r+1}} \exp\left(-(r-1)y^{-1}\ln(1-xy)\right) \exp\left(\exp\left(-y^{-1}\ln(1-xy)\right) - 1\right) \\
&= \frac{(1-xy)^{(1-r)/y}}{(1-xy)^{r+1}} \exp\left(\left(\frac{1}{1-xy}\right)^{1/y} - 1\right) = \frac{\exp\left((1-xy)^{-1/y} - 1\right)}{(1-xy)^{(r-1)/y+r+1}}.
\end{aligned}$$

□

Remark 2.1. *The distribution of the colored singletons statistic on members of $\mathcal{K}_n^{(r)}$ having a fixed number k of non-special blocks is seen to be given by*

$$\sum_{j=n-k}^{n} v_{j,k+j-n}^{(r)} c_{n,j}^{(r+1)} y^{n-j}.$$

However, the egf over $n \geq k$ for a fixed k does not seem to have a simple formula since one cannot separate the $v_{j,k+j-n}^{(r)}$ and $c_{n,j}^{(r+1)}$ terms when sums are interchanged (as happens with $v_j^{(r)}$ and $c_{n,j}^{(r+1)}$ in the preceding proof). In the case of a fixed number of colors, it is possible to compute the analogous egf formula (see Lemma 3.1 below).

Taking $y = 1$ in the preceding result, and recalling $\sum_{n \geq 0} L_n^{(r)} \frac{x^n}{n!} = \frac{1}{(1-x)^{2r}} \exp\left(\frac{x}{1-x}\right)$, yields the following connection between colored set partitions and Lah distributions.

Corollary 2.1. *If $n, r \geq 0$, then $|\mathcal{K}_n^{(r)}| = L_n^{(r)}$. In particular, the number of partitions of $[n]$ in which a singleton $\{x\}$ is colored in one of x ways for each $x \in [n]$ is given by L_n.*

Let $\mathcal{E}_n^{(r)}$ and $\mathcal{O}_n^{(r)}$ denote the subsets of $\mathcal{K}_n^{(r)}$ whose members contain an even or an odd number of singletons in I, respectively. Substituting $y = -1$ in Theorem 2.1 gives $G^{(r)}(x,-1) = \frac{e^x}{(1+x)^2}$, and extracting the coefficient of $x^n/n!$ yields the following sign-balance result.

Corollary 2.2. *If $n, r \geq 0$, then $|\mathcal{E}_n^{(r)}| - |\mathcal{O}_n^{(r)}| = \sum_{m=0}^{n} (-1)^m \binom{n}{m} (m+1)!$.*

Let $H_n = \sum_{i=1}^{n} \frac{1}{i}$ denote the n-th harmonic number for $n \geq 1$. We have the following explicit formula for the total number of singletons in I among all members of $\mathcal{K}_n^{(r)}$.

Theorem 2.2. *If $n \geq 1$ and $r \geq 0$, then the total number of colored singletons in all r-partitions of $[r+n]$ in which a singleton $\{x\}$ where $x \in [r+1, r+n]$ is colored in one of x ways is given by*

$$n(2r+1)L_{n-1}^{(r+1)} - rnL_{n-1}^{(r)} - 2rn(n-1)L_{n-2}^{(r+1)} - \sum_{m=2}^{n} \binom{n}{m} m! \left(H_{m-1} + \frac{r}{m}\right) L_{n-m}^{(r)}.$$

Proof. Differentiating formula (7) with respect to y gives

$$\begin{aligned}\frac{\partial G^{(r)}(x,y)}{\partial y} &= \exp\left((1-xy)^{-1/y} - 1\right)\left(\frac{1}{(1-xy)^{r/y+r+1}}\left(\frac{x}{y(1-xy)} + \frac{\ln(1-xy)}{y^2}\right) + \frac{(r-1)\ln(1-xy)}{y^2(1-xy)^{(r-1)/y+r+1}}\right.\\ &\quad \left. + \frac{x(r-1+(r+1)y)}{y(1-xy)^{(r-1)/y+r+2}}\right),\end{aligned}$$

and hence

$$\begin{aligned}\frac{\partial G^{(r)}(x,y)}{\partial y}\,|_{y=1} &= \exp\left(\frac{x}{1-x}\right)\left(\frac{1}{(1-x)^{2r+1}}\left(\frac{x}{1-x} + \ln(1-x)\right) + \frac{(r-1)\ln(1-x)}{(1-x)^{2r}} + \frac{2rx}{(1-x)^{2r+1}}\right)\\ &= \frac{\exp\left(\frac{x}{1-x}\right)}{(1-x)^{2r+2}}\left((2r+1)x - 2rx^2 + (1-x)(r-(r-1)x)\ln(1-x)\right)\\ &= \frac{\exp\left(\frac{x}{1-x}\right)}{(1-x)^{2r+2}}\left((2r+1)x - 2rx^2\right) - \frac{\exp\left(\frac{x}{1-x}\right)}{(1-x)^{2r}} \cdot \left(-\frac{\ln(1-x)}{1-x}\right) \cdot (r-(r-1)x)\,.\end{aligned}$$

Extracting the coefficient of $\frac{x^n}{n!}$ for $n \geq 2$, and recalling $\sum_{n\geq 1} H_n x^n = -\frac{\ln(1-x)}{1-x}$, gives

$$\begin{aligned}[x^n/n!]\left(\frac{\partial G^{(r)}(x,y)}{\partial y}\,|_{y=1}\right) &= n(2r+1)L_{n-1}^{(r+1)} - 2rn(n-1)L_{n-2}^{(r+1)} - n!r\sum_{m=1}^{n} H_m \frac{L_{n-m}^{(r)}}{(n-m)!}\\ &\quad + n!(r-1)\sum_{m=1}^{n-1} H_m \frac{L_{n-m-1}^{(r)}}{(n-m-1)!}\\ &= n(2r+1)L_{n-1}^{(r+1)} - 2rn(n-1)L_{n-2}^{(r+1)} - r\sum_{m=1}^{n}\binom{n}{m} m! H_m L_{n-m}^{(r)}\\ &\quad + (r-1)\sum_{m=1}^{n-1}\binom{n}{m+1}(m+1)! H_m L_{n-m-1}^{(r)}.\end{aligned}$$

The two sums in the preceding expression may be combined to give

$$\begin{aligned}&-r\sum_{m=1}^{n}\binom{n}{m} m! H_m L_{n-m}^{(r)} + (r-1)\sum_{m=2}^{n}\binom{n}{m} m! H_{m-1} L_{n-m}^{(r)}\\ &= -rnL_{n-1}^{(r)} - r\sum_{m=2}^{n}\binom{n}{m} m! \cdot \frac{1}{m} L_{n-m}^{(r)} - \sum_{m=2}^{n}\binom{n}{m} m! H_{m-1} L_{n-m}^{(r)}\\ &= -rnL_{n-1}^{(r)} - \sum_{m=2}^{n}\binom{n}{m} m! \left(H_{m-1} + \frac{r}{m}\right) L_{n-m}^{(r)},\end{aligned}$$

which implies the desired formula. □

From formula (6), it is seen that the total number of singletons in all the members of $\mathcal{K}_n^{(r)}$ is also given by the summation $\sum_{j=0}^{n}(n-j)v_j^{(r)} c_{n,j}^{(r+1)}$. Equating this with the prior result gives the following apparently new identity relating r-Stirling, r-Lah and harmonic numbers.

Corollary 2.3. *If $n \geq 1$ and $r \geq 0$, then*

$$\sum_{j=0}^{n}(n-j)v_j^{(r)} c_{n,j}^{(r+1)} = n(2r+1)L_{n-1}^{(r+1)} - rnL_{n-1}^{(r)} - 2rn(n-1)L_{n-2}^{(r+1)} - \sum_{m=2}^{n}\binom{n}{m} m! \left(H_{m-1} + \frac{r}{m}\right) L_{n-m}^{(r)}. \tag{8}$$

Let t_n denote the total number of non-special singletons in all the members of $\mathcal{K}_n^{(r)}$ for a fixed $n \geq 1$ and $r \geq 0$ variable. It is seen that t_n is a polynomial in r of degree n. The first several t_n are given in Table 1. Note that taking $r = 0$ in t_n yields the total number of singletons in all the members of $\mathcal{P}_n$ wherein each singleton $\{x\}$ receives one of x colors.

We now provide combinatorial explanations of the formulas found above for the cardinality of $\mathcal{K}_n^{(r)}$ and for the sign balance of the colored singletons statistic on $\mathcal{K}_n^{(r)}$.

n	t_n
1	$r+1$
2	$4r^2+9r+4$
3	$12r^3+48r^2+58r+24$
4	$32r^4+200r^3+444r^2+444r+176$
5	$80r^5+720r^4+2500r^3+4360r^2+3941r+1505$
6	$192r^6+2352r^5+11680r^4+30900r^3+47000r^2+39665r+14652$
7	$448r^7+7168r^6+48048r^5+178080r^4+400876r^3+556332r^2+445558r+159768$

Table 1: The polynomials t_n for $1 \le n \le 7$.

Combinatorial proof of Corollary 2.1

We shall represent the element x in a colored singleton within a member of $\mathcal{K}_n^{(r)}$ by x_y, where $y \in [x]$ denotes the color assigned to x. By a *block right-left minimum* within $\pi \in \mathcal{L}_n^{(r)}$, we mean an element z in some block B of π in which all elements occurring to the right of z within B are greater than z. That is, if the (ordered) contents of B is given by the sequence $i_1 \cdots i_p$, then the letter i_a corresponds to a block right-left minimum (rl min) if and only if $i_j > i_a$ for all $a < j \le p$.

We first define a mapping from $\mathcal{K}_n^{(r)}$ to $\mathcal{L}_n^{(r)}$ as follows. Given $\pi \in \mathcal{K}_n^{(r)}$, let P denote the set of singletons x_y in π such that $y \in [x-1]$ and let P' denote the partition of the members of $[r+n]-P$. If P is empty, then let $f(\pi)=\pi$, where we ignore the coloring of any singletons x_x in π. So assume P is nonempty with the members of P given by $i_1 < \cdots < i_\ell$ for some $\ell \ge 1$. Let the singleton $\{i_t\}$ for $t \in [\ell]$ be assigned the color j_t. We first insert i_1 into the partition P' such that it directly precedes the element j_1, but otherwise ignore the color assigned to i_1. Next, we insert i_2 into the resulting (contents-ordered) partition such that it directly precedes j_2, and proceed likewise with the elements $i_3, \ldots, i_\ell$, successively. Let $f(\pi)$ denote the member of $\mathcal{L}_n^{(r)}$ that results once all the members of P have been inserted as described and the coloring of any singletons x_x within π is disregarded.

Note that the inserted members of I from P correspond precisely to the set of elements that are not rl min within $f(\pi)$. This follows from the fact that each element i_t is inserted directly prior to a member of $[r+n]$ that is less than i_t and that such an insertion does not affect the status of any other rl min. Thus, the mapping f may be reversed as follows. If all the elements of $\sigma \in \mathcal{L}_n^{(r)}$ are rl min (i.e., if σ corresponds to a member of $\mathcal{P}_n^{(r)}$), then let $g(\sigma)=\sigma$, where any singletons $\{x\}$ within σ are assigned the color x. So assume at least one element of $[r+n]$ within σ does not correspond to an rl min and let $a_1 < \cdots < a_\ell$ denote the set of non rl min for some $\ell \ge 1$. First observe that $\{a_1, \ldots, a_\ell\} \subseteq [r+1, r+n]$ since members of $[r]$ occur in different blocks of π. Further, we have that a_ℓ must be followed directly by some member of $[a_\ell - 1]$ within its block. For if not, then a_ℓ not being the final element within any block implies it would be followed by some $z > a_\ell$. But then a_ℓ being the largest non rl min implies z must be an rl min. Thus, all of the elements to the right of z in its block must be greater than z, and hence a_ℓ. This would imply a_ℓ would be an rl min, which it isn't.

We then remove a_ℓ from its block within σ and form the singleton $\{a_\ell\}$, which we assign the color q, where $q \in [a_\ell - 1]$ denotes the successor of a_ℓ within its block. Note that moving a_ℓ as described does not create or destroy any rl min (in blocks other than the new singleton $\{a_\ell\}$), as the successor of a_ℓ in its block was smaller than a_ℓ. In the resulting partition where a_ℓ occurs as a colored singleton, we have that $a_{\ell-1}$ must be followed by a smaller element t in its block, upon reasoning as before. We then move $a_{\ell-1}$ from its block and create the singleton $\{a_{\ell-1}\}$, which is assigned the color t. We proceed likewise, successively, with $a_{\ell-2}, \ldots, a_1$. After a_1 has been moved and its singleton assigned some color in $[a_1 - 1]$, we assign to any uncolored singletons $\{x\}$ where $x \in I$ occurring within the partition at this point the color x. Let $g(\sigma)$ denote the resulting member of $\mathcal{K}_n^{(r)}$.

For example, if $n=12$, $r=3$,

$$\pi = \{1,7,8\},\ \{2,4\},\ \{3\},\ \{6,10,14\},\ \{5_4\},\ \{9_9\},\ \{11_5\},\ \{12_9\},\ \{13_3\},\ \{15_8\} \in \mathcal{K}_{12}^{(3)}$$

and

$$\sigma = \{1,7,15,8\},\ \{2,11,5,4\},\ \{13,3\},\ \{6,10,14\},\ \{12,9\} \in \mathcal{L}_{12}^{(3)},$$

then we have $f(\pi)=\sigma$ and $g(\sigma)=\pi$. One can then show $g(f(\pi))=\pi$ for all $\pi \in \mathcal{K}_n^{(r)}$ since g when applied to $f(\pi)$ is seen to restore each of the singleton blocks of π in I along with their respective colors. Likewise, $f(g(\sigma))=\sigma$ for all $\sigma \in \mathcal{L}_n^{(r)}$ since f when applied to $g(\sigma)$ sequentially recovers the non rl min of σ in the reverse order in which they were taken away by g. Thus, the mapping f provides a bijection between $\mathcal{K}_n^{(r)}$ and $\mathcal{L}_n^{(r)}$, whence $|\mathcal{K}_n^{(r)}| = L_n^{(r)}$. $\square$

Combinatorial proof of Corollary 2.2

We first show

$$|\mathcal{E}_n^{(r)}| - |\mathcal{O}_n^{(r)}| = (-1)^n \sum_{m=0}^{n} \binom{n}{m} m! d_{n-m}, \qquad n, r \geq 0, \tag{9}$$

where d_n denotes the number of derangements of $[n]$ (i.e., permutations without fixed points), see A000166 in [19]. We demonstrate first the $r = 0$ case of (9), where we may clearly assume $n \geq 2$. From the combinatorial proof of Corollary 2.1 above, one has that the statistic on $\mathcal{K}_n^{(0)}$ recording the number of singleton blocks has that same distribution as the statistic on $\mathcal{L}_n = \mathcal{L}_n^{(0)}$ which records the number of elements $p \in [n]$ within $\pi \in \mathcal{L}_n$ such that either (i) p is not a block rl min of π or (ii) p is both the last and smallest element of some block of π. Define the sign of π by $(-1)^{\sigma(\pi)}$, where $\sigma(\pi)$ denotes the number of elements of $[n]$ satisfying either condition (i) or (ii). It then suffices to define a sign-changing involution on $\mathcal{L}_n$ off of a subset of $\mathcal{L}_n$ whose members have sum of signs given by the right-hand side of (9).

To do so, we first order the blocks of $\pi \in \mathcal{L}_n$ from left to right in increasing order of their smallest elements contained therein. Consider a block B of π, if it exists, such that $|B| \geq 2$ whose last element (when considering the sequence of elements within B from left to right) is neither the smallest nor the second smallest element of B. Assume B is the leftmost block of π satisfying this requirement and let a, b with $a < b$ denote the two smallest elements of B. We then switch the elements a and b within B, leaving all other members of B undisturbed. Let $\upsilon(\pi)$ denote the resulting member of $\mathcal{L}_n$. Note that b changes its status concerning being an rl min of π, with no other members of $[n]$ changing in this regard. Further, since neither a nor b is the final element of B, we have that π and $\upsilon(\pi)$ are of opposite σ-parity. Thus, υ yields a sign-changing involution of $\mathcal{L}_n$ in all cases when it is defined. Note that if a were to occur at the end of B, then switching a and b would fail to reverse the parity since a and b would both go from contributing to the σ value of π to neither doing so, with all other elements of $[n]$ remaining of the same status concerning their contributing to $\sigma(\pi)$.

We now determine the sum of the signs of the members of the set T of survivors of υ, which is seen to consist of those π in which each non-singleton block B ends either in its smallest or second smallest element. Note that all elements in singleton blocks contribute to the value of $\sigma(\pi)$, with the same holding for non-singletons whose smallest element is last. If B is a non-singleton block whose second smallest letter is last, then the smallest two elements of that block do not contribute to $\sigma(\pi)$, with all other elements in B doing so (being non rl min). Thus, it is seen that each member of T has sign $(-1)^n$. To enumerate the members of T, let m denote the number of elements of $[n]$ going in either singletons or non-singletons whose smallest element is last. Once those members of $[n]$ have been selected, there are $m!$ ways in which to arrange them (as blocks in this case may be viewed as cycles of an arbitrary permutation of size m), with d_{n-m} ways in which to arrange the unselected members of $[n]$ as the remaining blocks are required to be non-singletons in which the second smallest element is last. Considering all possible $0 \leq m \leq n$ implies that the sum of the signs of the members of T is given by the right side of (9), which completes the proof in the $r = 0$ case.

We now show (9) in the case when $r > 0$. Recall that a special block within a member of $\mathcal{L}_n^{(r)}$ is one that contains a special element, i.e., a member of $[r]$. Note that from the proof of Corollary 2.1, we have that the colored singletons statistic on $\mathcal{K}_n^{(r)}$ where $r > 0$ corresponds to the statistic on $\mathcal{L}_n^{(r)}$ recording the number of $x \in I$ such that one of the following holds: (a) x occurs as a non rl min within a non-special block, (b) x occurs as both the smallest and last element of some non-special block, (c) x occurs as a non rl min in the sequence of elements to the right of the special element within some special block or (d) x occurs anywhere to the left of the special element within a special block. Let $\delta(\rho)$ denote the number of elements of I satisfying one of the conditions (a)–(d) above within $\rho \in \mathcal{L}_n^{(r)}$. If the sign of ρ is defined as $(-1)^{\delta(\rho)}$ for each ρ, then by the preceding the sum of signs of all members of $\mathcal{L}_n^{(r)}$ is given by the left-hand side of (9).

We now define an involution of $\mathcal{L}_n^{(r)}$ which reverses the δ-parity. If no elements of I occur in the special blocks of $\rho \in \mathcal{L}_n^{(r)}$, then we may apply the mapping υ defined above since only conditions (a) and (b) would apply. Otherwise, let s denote the smallest member of $[r]$ such that the s-th special block of ρ contains at least one member of I. Suppose first that at least two members of I occur to the right of s within its block S and let $u < v$ denote the smallest two such members. Then switching the positions of u and v is seen to change the parity since only the element v changes in regard to its contributing to the value of δ (as its status concerning condition (c) changes). Note here it is possible for either u or v to occur at the very end of S since u would not be counted by δ, regardless of its position. Now suppose there is at most one element of I occurring to the right of s within S. Let w denote the rightmost element of I occurring in S in this case. We then switch the letters s and w and observe that this changes the status of w with regard to its satisfying (d) above. Note further that w would not satisfy (c) when it occurs last in S since it would trivially be an rl min in this case. Therefore, each member of $\mathcal{L}_n^{(r)}$ for which at least one element of I occurs within a special block is paired with another of opposite δ-parity, which implies formula (9) holds for all $r > 0$ as well.

To complete the proof, we establish by a combinatorial argument the equality

$$\sum_{m=0}^{n}(-1)^{n-m}\binom{n}{m}(m+1)! = \sum_{m=0}^{n}\binom{n}{m}m!d_{n-m}, \qquad n \geq 0. \tag{10}$$

Consider the set $\mathcal{A}_{n,m}$ of ordered pairs (α, β), where α is a subset of $[2, n+1]$ of size $n-m$ and β is a permutation of $\{1\}\cup([2,n+1]-\alpha)$, arranged according to cycles. Define the sign of a member of $\mathcal{A}_{n,m}$ by $(-1)^{n-m}$ and let $\mathcal{A}_n = \cup_{m=0}^{n}\mathcal{A}_{n,m}$. Then the left side of (10) gives the sum of the signs of all members of $\mathcal{A}_n$. Define a sign-changing involution on $\mathcal{A}_n$ by identifying the smallest element of $[2, n+1]$ either belonging to α or occurring in β as a 1-cycle and switching it to the other option. The set of survivors of this involution consists of those ordered pairs (α, β) such that α is empty and β is a permutation of $[n+1]$ in which only the cycle containing 1 can have length one. Each of these (α, β) has positive sign, being a member of $\mathcal{A}_0$, and they are enumerated by the right side of (10), upon considering the number m of elements of $[2, n+1]$ appearing within the cycle of β containing 1. This establishes (10) and completes the proof of Corollary 2.2. □

3. Singletons with a fixed number of colors

Let $\mathcal{P}_{n,k}^{(r)}$ for $0 \leq k \leq n$ denote the subset of $\mathcal{P}_n^{(r)}$ whose members contain k non-special blocks (and hence $k+r$ blocks altogether). Let $S_{n,k}^{(r)}$ and $B_n^{(r)} = \sum_{k=0}^{n} S_{n,k}^{(r)}$ denote respectively the r-Stirling number of the second kind and r-Bell number (see, for example, [2] and [9]). Recall that the cardinalities of the sets $\mathcal{P}_{n,k}^{(r)}$ and $\mathcal{P}_n^{(r)}$ are given by $S_{n,k}^{(r)}$ and $B_n^{(r)}$, respectively.

Consider coloring singleton blocks of an r-partition using a fixed number of colors as follows. Given a positive integer s, let $\mathcal{P}_{n,k}^{(r,s)}$ denote colored members of $\mathcal{P}_{n,k}^{(r)}$ wherein each non-special singleton block (i.e., $\{x\}$ for $x \in I$) receives one of s colors, and let $\mathcal{P}_n^{(r,s)} = \cup_{k=0}^{n}\mathcal{P}_{n,k}^{(r,s)}$. Let $S_{n,k}^{(r,s)} = |\mathcal{P}_{n,k}^{(r,s)}|$ and $B_n^{(r,s)} = |\mathcal{P}_n^{(r,s)}|$.

Lemma 3.1. *For each $k \geq 0$, we have*

$$\sum_{k\geq 0} S_{n,k}^{(r,s)}\frac{x^n}{n!} = \frac{e^{rx}}{k!}\left(e^x + (s-1)x - 1\right)^k. \tag{11}$$

Proof. Considering the number j of non-special singletons within a member of $\mathcal{P}_{n,k}^{(r,s)}$ implies

$$S_{n,k}^{(r,s)} = \sum_{j=0}^{k}\binom{n}{j}v_{n-j,k-j}^{(r)}s^j, \qquad n \geq k \geq 0.$$

Computing the egf for a fixed $k \geq 0$ then gives

$$\begin{aligned}\sum_{n\geq k} S_{n,k}^{(r,s)}\frac{x^n}{n!} &= \sum_{n\geq k}\frac{x^n}{n!}\sum_{j=0}^{k}\binom{n}{j}v_{n-j,k-j}^{(r)}s^j = \sum_{j=0}^{k}\frac{(sx)^j}{j!}\sum_{n\geq k-j}v_{n,k-j}^{(r)}\frac{x^n}{n!} = \sum_{j=0}^{k}\frac{(sx)^j}{j!}\cdot\frac{e^{rx}}{(k-j)!}(e^x-x-1)^{k-j}\\ &= \frac{e^{rx}}{k!}\sum_{j=0}^{k}\binom{k}{j}(sx)^j(e^x-x-1)^{k-j} = \frac{e^{rx}}{k!}(e^x+(s-1)x-1)^k,\end{aligned}$$

where we have made use of the particular case of (2) where $x_i = y_i = 1$ for all $i > 1$, with $x_1 = 0$, $y_1 = 1$. □

Summing the formula in Lemma 3.1 over all $k \geq 0$, and using the fact $B_n^{(r,s)} = \sum_{k=0}^{n} S_{n,k}^{(r,s)}$, yields the following result.

Theorem 3.1. *We have $\sum_{n\geq 0} B_n^{(r,s)}\frac{x^n}{n!} = \exp\left(e^x + (r+s-1)x - 1\right)$ and hence $B_n^{(r,s+1)} = B_n^{(r+s)}$ for all n, r and s. In particular, the number of partitions of $[n]$ in which each singleton is colored in one of $s+1$ ways equals the number of s-partitions of $[s+n]$.*

The equality $B_n^{(r,s+1)} = B_n^{(r+s)}$ may be explained directly as follows. Let $\pi \in \mathcal{P}_n^{(r,s+1)}$ and denote the colors used on non-special singletons of π by elements in $[s+1]$. First consider the singletons of π assigned a color in $[s]$. We append s special blocks to π, each to contain a different element of $[r+1, r+s]$, and add s to all elements of I within π. Then add all non-special singletons of π which are assigned the color i to the special block containing $r+i$ for each $i \in [s]$. We leave any remaining colored singletons unchanged, which have been assigned the color $s+1$. Finally, remove the color from all colored elements and the resulting partition is seen to belong $\mathcal{P}_n^{(r+s)}$. Since the procedure just described is reversible, the equality in question is established.

Let $H^{(r)}(x, s)$ denote the egf formula for $B_n^{(r,s)}$ given in Theorem 3.1. Then $\frac{\partial}{\partial y}H^{(r)}(x, sy)\mid_{y=1} = sxH^{(r)}(x, s)$, and extracting the coefficient of $\frac{x^n}{n!}$ gives $nsB_{n-1}^{(r+s-1)}$. Thus, the total number of colored singletons within the members of $\mathcal{P}_n^{(r,s)}$ is given by $nsB_{n-1}^{(r+s-1)}$ for each $n \geq 1$, which may be realized directly. On the other hand, from the proof of Lemma 3.1, this number is also given by

$$\sum_{k=0}^{n}\sum_{j=0}^{k} j\binom{n}{j} v_{n-j,k-j}^{(r)} s^j = \sum_{j=1}^{n} j\binom{n}{j} v_{n-j}^{(r)} s^j = n\sum_{j=0}^{n-1}\binom{n-1}{j} v_{n-j-1}^{(r)} s^{j+1}.$$

Equating the two formulas for the total, and replacing n with $n+1$, gives the following identity.

Corollary 3.1. *If $n, r \geq 0$ and $s \geq 1$, then*

$$B_n^{(r+s-1)} = \sum_{j=0}^{n}\binom{n}{j} v_{n-j}^{(r)} s^j. \tag{12}$$

A combinatorial argument similar to that given above for Theorem 3.1 may be provided for Corollary 3.1. Let $\mathcal{E}_n^{(r,s)}$ and $\mathcal{O}_n^{(r,s)}$ denote the subsets of $\mathcal{P}_n^{(r,s)}$ whose members contain an even or an odd number of non-special singleton blocks, respectively. Define the r-Bell number $B_n^{(r)}$ for an arbitrary real number r as the coefficient of $\frac{x^n}{n!}$ in $\exp\left(e^x + rx - 1\right)$. Substituting $-s$ for s in the formula for $H^{(r)}(x,s)$ and extracting coefficients yields the following sign balance result.

Corollary 3.2. *If $n, r \geq 0$ and $s \geq 1$, then $|\mathcal{E}_n^{(r,s)}| - |\mathcal{O}_n^{(r,s)}| = B_n^{(r-s-1)}$.*

Combinatorial proof of Corollary 3.2

Since both sides of the equality are seen to be polynomials in r for a fixed n and s, it suffices to establish the result in the case when $r \geq s+1$. By the bijection used in the combinatorial explanation of Theorem 3.1 above, the left-hand side of the equality is equal to the sign balance of the statistic μ on $\mathcal{P}_n^{(r+s-1)}$ which records the number of elements of $J = [r+s, r+s+n-1]$ either going in one of the final $s-1$ special blocks or occurring as a singleton. Let $\pi \in \mathcal{P}_n^{(r+s-1)}$, where $r \geq s+1$. Suppose first that there exists at least one member of J in one of the final $2s-2$ special blocks of π, which we label as $B_1, \ldots, B_{2s-2}$ so that block B_i contains the special element $r-s+i+1$ for $1 \leq i \leq 2s-2$. Suppose j_0 is the smallest index $j \in [s-1]$ such that $B_j \cup B_{j+s-1}$ contains at least one member of J. Let ℓ denote the smallest member of J in $B_{j_0} \cup B_{j_0+s-1}$ and move ℓ either from B_{j_0} to B_{j_0+s-1} or vice versa.

Otherwise, suppose none of the blocks $B_1, \ldots, B_{2s-2}$ contain a member of J. In this case, let m denote the smallest member of J, if it exists, that either occurs as a singleton or lies in the special block of π containing $r-s+1$. We switch options concerning the position of m within π. One may verify that this operation, taken together with the preceding one, defines a sign-changing involution on $\mathcal{P}_n^{(r+s-1)}$. The set of survivors of this involution consists of those $\lambda \in \mathcal{P}_n^{(r+s-1)}$ in which no element of J occurs as a singleton block or lies in the final $2s-1$ special blocks of λ. Such λ are synonymous with the members of $\mathcal{P}_n^{(r-s)}$ containing no non-special singletons. Upon taking any non-special elements lying in the $(r-s)$-th special block and forming a singleton block for each one, deleting $r-s$ and subtracting 1 from each non-special element, we have that the set of survivors λ are synonymous with the members $\mathcal{P}_n^{(r-s-1)}$. Since each λ has positive sign, the sign balance of the statistic μ on $\mathcal{P}_n^{(r+s-1)}$ is given by $B_n^{(r-s-1)}$, which completes the proof. □

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